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WITH THEIR

SOLUTIONS.

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Mathematical Papers, &c.

Solved Questions.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

7382. (By Professor SYLVESTER, F.R.S.) — If p and q are relative primes, prove that the number of integers inferior to pq which cannot be resolved into parts (zeros admissible), multiples respectively of p and q, is

$$\tfrac{1}{2}(p-1)(q-1).$$

[If $p = 4$, $q = 7$, we have $\tfrac{1}{2}(p-1)(q-1) = 9$; and 1, 2, 3, 5, 6, 9, 10, 13, 17 are the only integers inferior to 28, which are neither multiples of 4 or 7, nor can be made up by adding together multiples of 4 and 7.]

Solution by W. J. CURRAN SHARP, M.A.

If the product $(1+x^p+x^{2p}+\ldots+x^{pq})(1+x^q+x^{2q}+\ldots+x^{pq})$ be considered, each term between 1 and x^{pq} corresponds to a number less than pq, and of the form $mp+nq$; also $2x^{pq}$ is the middle term, and the coefficients from each end are the same. Hence twice the number of integers of the form $mp+nq$, and less than pq, is the value of the above product when $x=1$ with four deducted, since the terms involving x^1, x^{pq}, x^{2pq} are not included; and therefore the number of these integers is

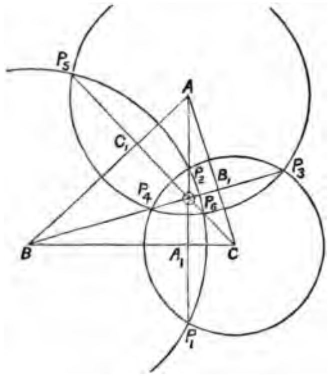
$$\tfrac{1}{2}(p+1)(q+1)-2,$$

and the number of those which cannot be put into this form

$$=pq-1-[\tfrac{1}{2}(p+1)(q+1)-2] = \tfrac{1}{2}[pq-p-q+1] = \tfrac{1}{2}(p-1)(q-1).$$

1585. (By the late Professor CLIFFORD, F.R.S.)—If three circles are mutually orthotomic, prove that the circles on their common chords as diameters have a common radical axis.

Solution by Asûtosh Mukhopâdhyây.



Let A, B, C be the centres of the three mutually orthotomic circles; $P_1, P_2, P_3, P_4, P_5, P_6$ their points of intersection. Then, from the ordinary theory of orthotomic circles, we know that the centre of each circle lies on the common chord of the other two: it can also be shown, from elementary geometry, that the line of centres of any two intersecting circles bisects their common chord at right angles; hence, we infer that, in the triangle ABC, AA_1, BB_1, CC_1 are the perpendiculars, and its orthocentre O is the radical centre of the system.

Moreover, calling the sides of the triangle ABC, a, b, c, as usual, we have

$$\left.\begin{array}{l} BA_1 = c\cos B \\ A_1C = b\cos C \end{array}\right\},\quad \left.\begin{array}{l} CB_1 = a\cos C \\ B_1A = c\cos A \end{array}\right\},\quad \left.\begin{array}{l} AC_1 = b\cos A \\ C_1B = a\cos B \end{array}\right\}.$$

Now, let us call S_1, S_2, S_3 the circles described on P_1P_2, P_3P_4, P_5P_6 as diameters; let their radii be ρ_1, ρ_2, ρ_3. Then we shall have

$$\rho_1^2 = A_1P_1^2 = BA_1 \,.\, A_1C \text{ (because, the circles being orthotomic,}$$
$$= bc\cos B\cos C. \qquad \angle BP_1C \text{ is a right angle)}$$

Similarly, $\rho_2^2 = AB_1 \,.\, B_1C = ca \,.\, \cos C \,.\, \cos A,$

and $\rho_3^2 = AC_1 \,.\, C_1B = ab \,.\, \cos A \,.\, \cos B.$

Taking A_1 as origin, A_1C, A_1A as the rectangular axes of x and y, the coordinates of the points A_1, B_1, C_1 will be (0, 0), ($c\cos A\cos C$, $a\cos C\sin C$), (*$b\cos A\cos B$, $a\cos B\sin B$), respectively. But, A_1, B_1, C_1 are the centres of the circles S_1, S_2, S_3 of radii ρ_1, ρ_2, ρ_3, therefore the equations of these circles are

$$x^2+y^2 = bc\cos B\cos C \quad\ldots\ldots(1),$$

$$(x-c\cos A\cos C)^2+(y-a\cos C\sin C)^2 = ca\cos C\cos A \quad\ldots\ldots(2),$$

$$(x-b\cos A\cos B)^2+(y-a\cos B\sin B)^2 = ab\cos A\cos B \quad\ldots\ldots(3).$$

Subtracting (2) from (1), attending to the relation $\sin A : a =$ &c., and cancelling like terms, we get, for the equation of the radical axis,

$$x\cos A + y\sin A = \tfrac{1}{2}(c\cos C + b\cos B - a\cos A) \quad\ldots\ldots(4).$$

Subtracting (3) from (1), we get the same equation as (4). Therefore, the three circles have a common radical axis.

[This Question has been discovered by Mr. Walker to be erroneous. In fact, "solvitur delineando": if we draw (even mentally) the circles on P_1P_2, ... as diameters, we see they cannot have a common radical axis. The error of the above solution lies in the sign of the abscissa of C_1 (at * above) which should be *negative*, the abscissa being $-b\cos A\cos B$. The equations of the three radical axes are in fact

$$\pm x\cos A + y\sin A = \tfrac{1}{2}(-a\cos A + b\cos B + c\cos C)$$

and $(b\cos B + c\cos C)\cos A x + (-b\cos B + c\cos C)\sin A y$
$= (-b\cos B + c\cos C)(-a\cos A + b\cos B + c\cos C)$.

The three have the same radical centre as the three orthogonal circles, viz. the point O. The common chords of the three derived circles are equal; in fact, the square of each is

$$2(a^2+b^2+c^2)\cos A\cos B\cos C = 2(b'c'+c'a'+a'b') - (a'^2+b'^2+c'^2),$$

where $a' = B_1 C_1 \ldots$, and the three chords are parallel respectively to the three radii drawn from the circum-centre of A, B, C to its corners. The subject is re-proposed for further discussion as Question 7605.]

7455 & 7481. (By Professor TOWNSEND, F.R.S.)—(7455). A system of plane waves, propagated by rectilinear vibrations perpendicular to the plane incidence, being supposed divided into two by refraction and reflexion at the surface of separation of two isotropic elastic solids in molecular contact with each other; determine, given the coefficients of resistance to compression and to distortion for both solids,—

(*a*) The relative amplitudes of vibration, for any angle of incidence, of the three systems of waves.

(*b*) The particular angle of incidence corresponding to the evanescence of the reflected vibrations.

(7481.) A system of plane waves, propagated either by normal or by transversal vibrations, being supposed divided into two by perpendicular refraction and reflexion at the surface of separation of two isotropic elastic solids in molecular contact with each other; show that, in either case, the *vis viva* of the original is divided without loss of total amount between the two derived systems of waves.

Solution by the PROPOSER.

(7455.) Denoting by k, k', k_1 the amplitudes of the incident reflected and refracted vibrations, by θ, θ', θ_1 the angles of incidence, reflexion and refraction of the wave systems, and by μ, μ', μ_1 and ν, ν', ν_1 the coefficients of resistance to compression and to distortion respectively in their propagation through the solids; then, the vibrations being manifestly perpendicular to the plane of incidence, and therefore to the direction of propagation in each derived as well as in the original system of waves, and all systems of waves propagated by transversal vibrations in isotropic media producing only distortion, but not compression of the molecules of the media, the six ordinary equations of condition, three geometrical and three dynamical, at the separating surface of the solids become reduced in this simple case to the two, one geometrical and one dynamical, viz., $k+k'=k_1$ and $\nu\cot\theta k + \nu'\cot\theta' k' = \nu_1\cot\theta_1 k_1$, or, since in the same case $\nu'=\nu$ and $\cot\theta' = -\cot\theta$, to the equivalent two $(k+k') = k_1$ and $\nu\cot\theta(k-k') = \nu_1\cot\theta_1 k_1$; from which, solving for k' and k_1, we get at once that $k' = k\dfrac{\nu\cot\theta - \nu_1\cot\theta_1}{\nu\cot\theta+\nu_1\cot\theta_1}$, and that $k_1 = k\dfrac{2\nu\cot\theta}{\nu\cot\theta+\nu_1\cot\theta_1}$, which giving in all cases the ratios of k' and k_1 to k in terms of θ and θ_1 and of ν and ν_1, and showing that $k' = 0$ when $\nu_1 \cot \theta_1 = \nu \cot \theta$, supply in consequence the solutions of both parts of the question.

The angle of incidence θ for which $k' = 0$, with the corresponding angle of refraction θ_1 for which $k_1 = 1$, may be readily determined, in terms of the coefficients ν and ν_1, and of the densities ρ and ρ_1 of the two solids, as follows. Since, in that case, as appears from the above, $\nu \cot \theta = \nu_1 \cot \theta_1$, and since in all cases, from the known laws of wave refraction,

$$\sin \theta : \sin \theta' = v : v' = \left(\frac{\nu}{\rho}\right)^{\frac{1}{2}} : \left(\frac{\nu_1}{\rho_1}\right)^{\frac{1}{2}},$$

where v and v' are the velocities of propagation for transversal vibrations in the solids, therefore here

$$(\rho\nu)^{\frac{1}{2}} \cdot \cos \theta = (\rho_1 \nu_1)^{\frac{1}{2}} \cdot \cos \theta' \text{ and } (\rho\nu_1)^{\frac{1}{2}} \cdot \sin \theta = (\rho_1 \nu)^{\frac{1}{2}} \cdot \sin \theta_1 ;$$

from which, by elimination successively of θ_1 and θ, it appears at once that

$$\tan^2 \theta = \frac{\nu}{\nu_1} \cdot \frac{\nu\rho - \nu_1\rho_1}{\nu\rho_1 - \nu_1\rho}, \text{ and that } \tan^2 \theta_1 = \frac{\nu_1}{\nu} \cdot \frac{\nu\rho - \nu_1\rho_1}{\nu\rho_1 - \nu_1\rho},$$

which give manifestly the values of the two angles in terms of the four quantities in question.

Multiplying together the two pairs of equivalents in the two general equations of condition at the separating surface of the media, viz., $(k + k') = k_1$, and $\nu \cot \theta\, (k - k') = \nu_1 \cot \theta_1 k_1$, we get at once the additional equation

$$\nu \cot \theta\, (k^2 - k'^2) = \nu_1 \cot \theta_1 k_1^2 ;$$

or, remembering that $\nu : \nu_1 = \rho v^2 : \rho_1 v_1^2 = \rho \sin^2 \theta : \rho_1 \sin^2 \theta_1$, the equivalent equation $\rho \sin \theta \cos \theta\, (k^2 - k'^2) = \rho_1 \sin \theta_1 \cos \theta_1 k_1^2$; from which it appears, by the usual mode of inference, that the *vis viva* of the entire motion is the same after as before the division of the original wave system into two at the separating surface of the media: a result technically termed *the preservation of the vis viva* in the division at the surface.

(7481.) For perpendicular incidence, either for transversal or for normal vibrations, the two equations of condition at the separating surface of the media assume alike the same simplified forms, viz.,

$$(k + k') = k_1 \text{ and } \rho v\, (k - k') = \rho_1 v_1 k_1 ;$$

where v and v_1 are the velocities of propagation corresponding to the species of vibration, whichever it be, in the media, and equal consequently to $\left(\frac{\nu}{\rho}\right)^{\frac{1}{2}}$ and $\left(\frac{\nu_1}{\rho_1}\right)^{\frac{1}{2}}$ for transversal, and to $\left(\frac{\mu + \frac{4}{3}\nu}{\rho}\right)^{\frac{1}{2}}$ and $\left(\frac{\mu_1 + \frac{4}{3}\nu_1}{\rho_1}\right)^{\frac{1}{2}}$ for normal vibrations. Therefore, for that incidence, in both cases alike, as appears at once by multiplication together of the two pairs of equivalents, $\rho v\, (k^2 - k'^2) = \rho_1 v_1 k_1^2$; and therefore, &c., as regards the property in question.

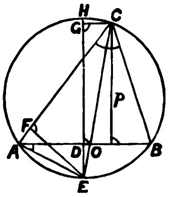
Solving for k' and k_1 from the two equations of condition, we see also that, in both cases alike,

$$k' = k \frac{\rho v - \rho_1 v_1}{\rho v + \rho_1 v_1} \text{ and } k_1 = k \frac{2\rho v}{\rho v + \rho_1 v_1},$$

from which it follows at once that, in both cases alike, $k' = 0$ and $k_1 = k$ when the products ρv and $\rho_1 v_1$ are equal in the media.

7357. (By Professor A. MOREL.)—Résoudre un triangle, connaissant une hauteur, le rayon du cercle inscrit, et le rayon du cercle circonscrit.

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.

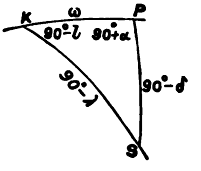


Let ABC be the triangle, then, denoting by a, b, c, s, R, r, p the three sides of the triangle, its semi-perimeter, the radii of its circumscribed and inscribed circles, and the altitude passing through the vertex C, we have $pc = \text{area} = rs$, therefore $s : c = r : p$ is known, and in the annexed well-known figure the ratio of CF $[= \frac{1}{2}(a+b)]$ to AD $(= \frac{1}{2}c)$ is known, or its equivalent CE : AE is known, and therefore $CE^2 : AE^2$ is known; but $CE . EO = AE^2$, therefore CE : AE = AE : EO, $\because$ CE : EO = $CE^2 : AE^2$ is known, and therefore CE : CO or EG : p is known, therefore EG is known, and ED, which is equal to EG $-p$, is known. Draw, then, in the given circumscribed circle a diameter EH, take ED as found above, through D draw AB perpendicular to EH, take DG $= p$, draw GC parallel to AB, then ACB is the required triangle.

7510. (By Professor HAUGHTON, F.R.S.)—If α, δ denote right ascension and declination, and l, λ longitude and latitude; show that the inclination of the ecliptic is given by the equation

$$\cos\omega = \frac{\sin\lambda\tan l + \sin\delta\tan\alpha}{\sin\lambda\tan\alpha + \sin\delta\tan l}.$$

Solution by B. REYNOLDS, M.A.; Professor MATZ, M.A.; *and others*.



By twice applying the cot.-formula to the triangle here shown (avoiding the angle of position S), we get

$$\tan\lambda\sin\omega - \sin l\cos\omega = -\tan\alpha\cos l \quad \ldots(1),$$

$$\tan\delta\sin\omega + \sin\alpha\cos\omega = \tan l\cos\alpha \quad \ldots\ldots(2);$$

whence, eliminating $\sin\omega$, we find

$$\cos\omega = \frac{\tan\lambda\tan l\cos\alpha + \tan\delta\tan\alpha\cos l}{\tan\lambda\sin\alpha + \tan\delta\sin l}$$

$$= \frac{\sin\lambda\tan l\dfrac{\cos\alpha}{\cos\lambda} + \sin\delta\tan\alpha\dfrac{\cos l}{\cos\delta}}{\sin\lambda\tan\alpha\dfrac{\cos\alpha}{\cos\lambda} + \sin\delta\tan l\dfrac{\cos l}{\cos\delta}} = \frac{\sin\lambda\tan l + \sin\delta\tan\alpha}{\sin\lambda\tan\alpha + \sin\delta\tan l},$$

since $\dfrac{\cos\alpha}{\cos\lambda} = \dfrac{\cos l}{\cos\delta}$, by a well-known formula.

"SYMMEDIANS" AND THE "T. R." CIRCLE. By R. TUCKER, M.A.

M. MAURICE D'OCAGNE, in the *Nouvelles Annales* for October, 1883, (pp. 450—464,) has applied the name of "Symmedian" to the line AS, in a triangle, which makes the same angle with AC that the median line AM does with AB (take the points, to fix the figure, which is easily drawn, in the order B, M, S, C). On AC take $AB' = AB$, and on AB, $AC' = AC$, then the "Symmedian" through A (say S_a) bisects $B'C'$. From M, S, let fall perpendiculars MP, SU on AB and MQ, SV on AC,

then $\quad SU/AS = MQ/AM$ and $SV/AS = MP/AM$,

therefore $\quad SU/SV = MQ/MP = AB/AC.$

Now $\quad SU \cdot AB/SV \cdot AC = SB/SC, \quad \therefore SB/SC = AB^2/AC^2$.......(1).

From (1) it readily follows that S_a, S_b, S_c pass through a point (P say). It is evident from (1) that P is the point through which the lines DPE′, EPF′, FPD′ must be drawn parallel to AB, BC, CA to obtain the points of section by the "T. R." circle (see *Educational Times* for July). The point P might be called the "Symmedian" point; its determination is easy from the above definition. Many interesting results are obtained by M. d'Ocagne.

It is also worth remarking that the "Symmedian" point P is the radical centre of the circles about ABD′E, BCE′F, CAF′D.

7461. (By Professor GENESE, M.A.)—A plane triangle is constructed whose sides are arcs of equal circles. If these sides be measured by the angles which they subtend at the centres of the corresponding circles, prove geometrically that (as an extension of Euc. I. 32), with a certain convention of signs, $A + B + C = \pi + a + b + c$.

Solution by the PROPOSER.

The result in the question was obtained thus. Conceive the whole triangle, supposing its sides concave to the interior, rotated about the centre of BC through the angle a so that B comes upon C, then rotated about C, in the same sense, through the angle $\pi - C$, so that the old arc BC now lies along CA. It is clear that, if the process be continued as suggested, the triangle will ultimately come back to its old position, that is, will have turned through the angle 2π. Thus we have

$$a + \pi - C + b + \pi - A + c + \pi - B = 2\pi \text{ or } \pi + a + b + c = A + B + C.$$

The cases wherein sides are convex to the interior may be similarly treated, or may be deduced from the above by producing the arcs so as to get a triangle of the first case. It is found that for convex sides a negative value must be given to a, b, c.

[This Question is almost identical with Mr. HEPPEL's Quest. 6214, solved by Mr. WALKER on p. 58 of Vol. 34 of the *Reprints*.]

7487. (By Professor WOLSTENHOLME, M.A., Sc.D.)—Given two conics U, U′, a tangent at P to U meets the polar of P with respect to U′ in P′; prove that the locus of P′ is the quartic $UV = U'^2$, where V is the polar reciprocal of U with respect to U′ so taken that the discriminants of U, U′, V are in geometrical progression.

Solution by T. WOODCOCK, B.A.; G. B. MATHEWS, B.A.; *and others.*

Using areal coordinates, we may write U and U′ in the forms

$$x^2+y^2+z^2=0 \text{ ....(1)},$$

and $ax^2+by^2+cz^2=0$; the equation to V will be $a^2x^2+b^2y^2+c^2z^2=0$. We have to eliminate xyz between (1) and the pair $x\xi+y\eta+z\zeta=0$, $ax\xi+by\eta+cz\zeta=0$. From the last two, we have

$$\frac{x}{(b-c)\,\eta\zeta}=\frac{y}{(c-a)\,\zeta\xi}=\frac{z}{(a-b)\,\xi\eta};$$

therefore, by (1), $\quad \eta^2\zeta^2(b^2+c^2-2bc)+\ldots+\ldots=0.$

This may be put in the form $(\xi^2+\eta^2+\zeta^2)(a^2\xi^2+\ldots+\ldots)=(a\xi^2+\ldots+\ldots)^2$, which is the one required.

7525. (By T. MUIR, M.A., F.R.S.E.)—If in a determinant of the n^{th} order the elements in the main diagonal be all negative and all the others positive, prove that the number of positive terms in the development is $\qquad \frac{1}{2}n!-(-2)^{n-2}(n-2).$

Solution by the PROPOSER; R. LACHLAN, B.A.; *and others.*

Let x, y be the numbers of positive and negative terms, respectively, and D the value of the determinant in question when each element is in magnitude equal to unity; then we have

$$x-y=D=(-2)^{n-1}(n-2) \text{ and } x+y=n!\text{; whence, \&c.}$$

7470. (By J. HAMMOND, M.A.)—Trace the curve

$$a^2+(b-r)^2-2a(b-r)\cos\theta=c^2,$$

with special reference to the cases (1) when $a^2+b^2=c^2$, (2) when $a\pm b\pm c=0$; and prove that it admits of an easy mechanical description.

[When $c=a$, the curve degenerates into a circle and a limaçon.]

Solution by G. B. Mathews, B.A.; Prof. Nash, M.A.; *and others.*

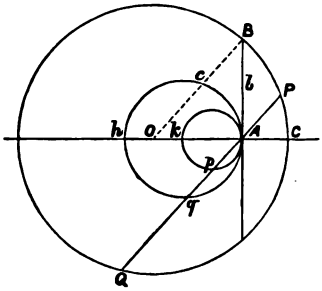


Fig. 1.

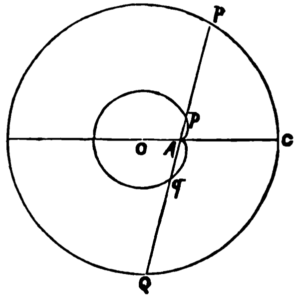


Fig. 2.

The construction throughout is to draw a chord PAQ of the fixed circle, and set off $Pp = Qq = b$. Then, if $OA = a$, $OC = c$, $AP = r$, $CAp = \theta$, we have

$$c^2 = OP^2 = OA^2 + AP^2 - 2OA \,.\, AP \cos OA$$
$$= (b-r)^2 + a^2 - 2a(b-r)\cos\theta.$$

The first figure applies when $a^2 + b^2 = c^2$, the second when $c = a + b$, and so for each of the other cases.

7421. (By R. Knowles, B.A.)—Two equal tangents OP, OQ are drawn to a parabola; prove that (1) the angle POQ is bisected by the axis, and (2) the distance of the centre of the circle OPQ from the vertex is constant and equal to one-half the latus rectum.

Solution by A. Martin, B.A.; Kate Gale; *and others.*

It is clear that the point O must be on the axis; hence POQ is bisected by the axis, and, as $SO = SP = SQ$, the centre of the circle must be the focus.

7534. (By the Rev. T. C. Simmons, M.A.)—A number is known to consist of four digits whose sum is 10; show that the odds are 154 : 65 in favour of the sum of the digits of twice the number being equal to 11.

Solutions by B. Reynolds, M.A.; A. Martin, B.A.; *and others.*

Of 4-digit numbers, those are favourable to the event specified which have *one* digit equal to or greater than 5. Unfavourable ones are those with two 5's (and two 0's therewith, of course), or with all the digits less than 5; hence, considering the numbers according to the highest digit in each, and remembering that 0 cannot stand as a first digit, the cases for and against are as follows:—

9100, 8200, 7300, 6400 give 6 each	= 24
8110, 6220 give 9 each	= 18
7111 gives	= 4
7210, 6310, 5410, 5320 give 18 each	= 72
6211, 5311, 5221 give 12 each..................	= 36
Total of favourable cases.........	=154
5500 gives	= 3
4222, 3331 give 4 each	= 8
4411 gives	= 6
4420 and 4330 give 9 each	= 18
4321 gives...	= 24
3322 gives...	= 6
Total of unfavourable cases......	= 65

7526. (By W. G. Lax, B.A.)—A swing-bridge is movable about a vertical axis on one bank of a river, and has a load of ballast suspended from the tail end of it; if the cost of bridge per ton be n times that of ballast, and the river a yards wide, find the length of the tail end of the bridge so that the cost of the whole may be a minimum.

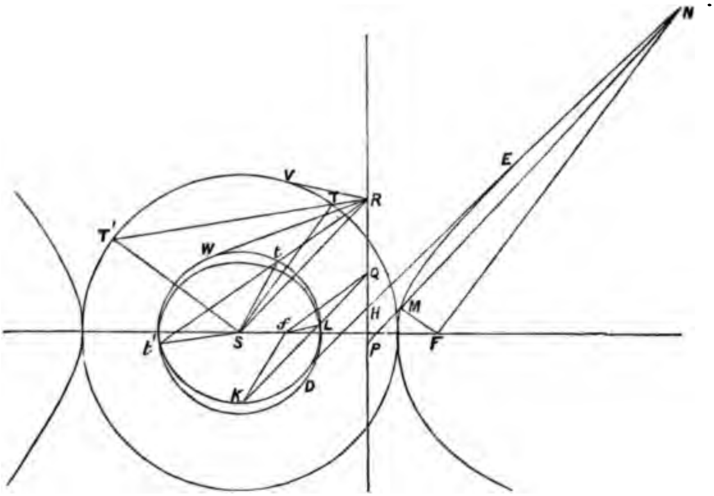
Solution by the Rev. T. C. Simmons, M.A.; Belle Easton; *and others.*

Let w = weight in tons of unit length, b = weight of ballast, x = required length of tail; then $\frac{1}{2}a \cdot wa = \frac{1}{2}x \cdot wx + x \cdot b$, whence $b = \frac{wa^2}{2x} - \frac{1}{2}wx$, and we require the minimum of $y \equiv nw(a+x) + \frac{wa^2}{2x} - \frac{1}{2}wx$. Putting $\frac{dy}{dx} = 0$, we get $nw - \frac{wa^2}{2x^2} - \frac{1}{2}w = 0$, whence $x^2 = \frac{a^2}{2n-1}$.

5350. (By S. A. Renshaw.)—An ellipse and hyperbola have the same centre and directrices, and they have a common tangent which touches the ellipse in D and the hyperbola in E, and meets one of the directrices in H. Also from the common centre of the curves S′R is drawn parallel to the common tangent and meeting the same directrix in R. Tangents RW, RV are drawn to the auxiliary circles of the ellipse and hyperbola. Show that, if FH, fH be joined, F and f being the foci of the curves belonging to the directrix RH,

$$DH \cdot HF : EH \cdot fH = WR' : VR.$$

Solution by the PROPOSER.



Draw any two parallels to DE, one cutting the ellipse in K, L and meeting the directrix in Q, the other cutting the hyperbola in M, N and meeting the directrix in P. Join PF, Qf, and draw RTT', Rtt' parallel to them, and meeting the auxiliary circles in T, T', t, t'. Then, since the auxiliary circles are generating circles of the curve, we have by similar triangles

$KQ \cdot QL : fQ^2 = tR \cdot Rt' : SR^2$ and $PF^2 : PM \cdot PN = SR^2 : TR \cdot RT'$;

therefore $KQ \cdot QL \cdot PF^2 : fQ^2 \cdot PM \cdot PN = tR \cdot Rt' : TR \cdot RT' = RW^2 : RV^2$;

when therefore the parallels both coincide with the tangent, this becomes

$DH^2 \cdot PF^2 : fQ^2 \cdot HE^2 = RW^2 : RV^2$ or $DH \cdot PF : fQ \cdot HE = RW : RV$.

7497. (By G. HEPPEL, M.A.)—If four concurrent normals meet an ellipse in points whose eccentric angles are α, β, γ, δ; show that $\alpha+\beta+\gamma+\delta = 3\pi$ or 5π, according as the ordinate of the point of concurrence is $-$ or $+$.

Solution by the Rev. J. L. KITCHIN, M.A.; Prof. MATZ, M.A.; *and others.*

The equation to the normal at the point whose eccentric angle is θ is

$$y - \frac{a}{b}\tan\theta \, . \, x = \frac{(a^2-b^2)\dfrac{a}{b}\tan\theta}{a\,(1+\tan^2\theta)^{\frac{1}{2}}},$$

therefore
$$y^2 - \frac{2axy}{b}\tan\theta + \frac{a^2x^2}{b^2}\tan^2\theta = \frac{(a^2-b^2)^2\tan^2\theta}{b^2(1+\tan^2\theta)};$$

whence $a^2x^2\tan^4\theta - 2\,abxy\tan^3\theta$

$$+\left[a^2x^2+b^2y^2-(a^2-b^2)^2\right]\tan^2\theta - 2abxy\tan\theta + b^2y^2 = 0.$$

Whence, since coefficients of $\tan^3\theta$ and $\tan\theta$ are equal and of same sign,

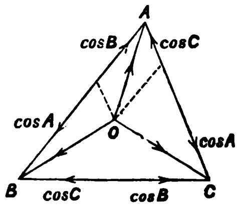
$$\tan\alpha+\tan\beta+\tan\gamma+\tan\delta = \tan\alpha\tan\beta\tan\gamma+\tan\alpha\tan\gamma\tan\delta+\dots;$$

whence $$\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}+\frac{\tan\gamma+\tan\delta}{1-\tan\gamma\tan\delta}=0,$$

or $\tan(\alpha+\beta)+\tan(\gamma+\delta)=0$; whence generally $\alpha+\beta+\gamma+\delta=n\pi$; n an integer. Now, since the last term is positive, only two, or four, can be negative together; this consideration with the geometry of the figure will show that n can be only 3 or 5.

7488. (By Professor Hudson, M.A.)—If O be the circumcentre of ABC, and forces act along OA, OB, OC proportional to BC, CA, AB; prove that their resultant passes through the in-centre.

Solution by C. Morgan, B.A., R.N.; G. B. Mathews, B.A.; *and others.*



The force along OA is equivalent to two forces proportional to cos B, cos C along BA, CA respectively; hence, resolving the other forces similarly as in the figure, the trilinear equation of the resultant is

$$(\cos B-\cos C)\,\alpha+(\cos C-\cos A)\,\beta + (\cos A-\cos B)\,\gamma = 0,$$

which goes through $\alpha=\beta=\gamma$; therefore &c.

5787 & 5945. (By W. J. C. Sharp, M.A.)—From an ordinary point on a quartic five straight lines can be drawn so as to be cut harmonically by two curves. How far is this modified when the point is a node?

(5945.) From a double point on a quintic, a triple point on a sextic, or a p^{ic} point on a $(p+3)^{ic}$, prove that a limited number of lines can be drawn so as to be harmonically cut by the curve. (This is an extension of Question 5787, which may be extended to surfaces as follows):—Through an ordinary point on a quartic surface lines may be drawn so as to be cut harmonically by the surface; the points of section will trace out a quintic curve on the surface.

Solution by the Proposer.

The $(n-p)^{\text{th}}$ polar of O with respect to an n-ic curve or surface is the locus of a point R in OR_n, cutting the figure in R_1, R_2, $R_3 \dots R_n$, such that $$\Sigma\,(OR_1-OR)\dots(OR_p-OR)\,OR_{p+1}\dots OR_n = 0.$$

Now, if R_1 coincide with O, and $OR_1 = 0$, this becomes

$$OR\Sigma (OR_2 - OR) \dots (OR_p - OR)\, OR_{p+1} \dots OR_n = 0,$$

and one value of OR is zero, as it should be.

If R_2 also coincide with O, the equation reduces to

$$OR^2\Sigma (OR_3 - OR) \dots (OR_p - OR)\, OR_{p+1} \dots OR_n = 0, \text{ and so on.}$$

Hence the 2nd polar of an n-ic, if $R_1, R_2 \dots R_{n-3}$ all coincide with O, has for its equation $OR^{n-3}[(OR_{n-2} - OR)\, OR_{n-1} \,.\, OR_n$

$$+ (OR_{n-1} - OR)\, OR_n \,.\, OR_{n-2} + (OR_n - OR)\, OR_{n-1} OR_{n-2}] = 0,$$

and $n-3$ values of OR are zero, and the other satisfies

$$\frac{3}{OR} = \frac{1}{OR_{n-2}} + \frac{1}{OR_{n-1}} + \frac{1}{OR_n};$$

and hence, if R coincide with R_{n-2}, R_{n-1}, or R_n, the line is divided harmonically; *i.e.*, if the line pass through an $(n-3)^{\text{ic}}$ point in the locus, and through an intersection of the locus and the second polar of the point, it is harmonically divided by the n-ic.

Now (1) a quartic curve is cut by the polar conic of a point on itself in five points, distinct from the point, the lines joining which to the original point are harmonically divided by the curve.

(2.) A quintic is cut by the polar cubic of a double point on it in $3 \times 5 - 4 \times 2 = 7$ points; and so many lines can be drawn through it so as to be harmonically divided by the curve, and generally a $(p+3)^{\text{ic}}$ is cut by the polar $(p+1)^{\text{ic}}$ of a p^{ic} point on it in $(p+1)(p+3) - (p+2)\,p = 2p+3$ points, distinct from the multiple point, and hence as many lines can be drawn through so as to be harmonically divided by the curve.

(3.) Since an n-ic surface meets any plane in an n-ic curve, the above argument shows that the curve of intersection of the second polar of an $(n-3)^{\text{ic}}$ point on it pierces the plane in $2n - 6 + 3 = 2n - 3$ points, and therefore this partial intersection is a $(2n-3)^{\text{ic}}$ curve; and therefore a quintic when the point is an ordinary point on a quartic surface.

5754. (By J. Hammond, M.A.)—Sum the series

$$\frac{1}{n} \cdot \frac{1}{2m+n} - 2m \cdot \frac{1}{n+1} \cdot \frac{1}{2m+n-1} + \frac{2m(2m-1)}{1\,.\,2} \cdot \frac{1}{n+2} \cdot \frac{1}{2m+n-2} - \&c.,$$

where m is a positive integer, and the $(r+1)^{\text{th}}$ term is

$$(-)^r \frac{2m(2m-1)\dots(2m-r+1)}{1\,.\,2\,.\,3 \dots r} \cdot \frac{1}{n+r} \cdot \frac{1}{2m+n-r}.$$

Solution by W. J. C. Sharp, M.A.; Rev. J. L. Kitchin, M.A.; *and others.*

Since $$\frac{1}{n+r} \cdot \frac{1}{2m+n-r} = \frac{1}{2(m+n)} \cdot \left(\frac{1}{n+r} + \frac{1}{2m+n-r}\right),$$

the proposed series may be written

$$\frac{1}{2(m+n)} \left[\frac{1}{n} - 2m \cdot \frac{1}{n+1} + \&c. + \frac{1}{2m+n} - 2m \cdot \frac{1}{2m+n-1} + \&c.\right];$$

$$\therefore \text{ Sum } = \frac{1}{m+n}\left\{\frac{1}{n} - 2m \cdot \frac{1}{n+1} + \&\text{c.}\right\} = \frac{1}{m+n}(1-\text{D})^{2m}\left(\frac{1}{n}\right),$$

using the notation of Boole's *Finite Differences*, p. 17,

$$= \frac{1}{m+n}(-\Delta)^{2m}\left(\frac{1}{n}\right) = \frac{(-1)^{2m}}{m+n}\Delta^{2m}\left(\frac{1}{n}\right) = \frac{1}{m+n} \cdot \frac{(-1)\ \ 2m\,!}{n(n+1)\dots n+2m}.$$

[The two bracketed series in line 4 are identical, and if we take twice the second, instead of twice the first, line 5 becomes (D being Boole's E)

$$\frac{1}{m+n}\left[\frac{1}{2m+n} - 2m\frac{1}{2m+n-1} + \&\text{c.}\right]$$

$$= \frac{1}{m+n}(\text{D}-1)^{2m}\left(\frac{1}{n}\right) = \frac{1}{m+n}\Delta^{2m}\left(\frac{1}{n}\right) = \&\text{c.}]$$

7489. (By Professor Wolstenholme, M.A., Sc D.)—Prove that, if $2s = \alpha+\beta+\gamma+\delta$, the equation of the directrix of the parabola that touches the four tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points whose excentric angles are $\alpha, \beta, \gamma, \delta$, is

$$\frac{x}{a}\left[\cos(s-\alpha) + \cos(s-\beta) + \cos(s-\gamma) + \cos(s-\delta)\right]$$

$$+ \frac{y}{b}\left[\sin(s-\alpha) + \sin(s-\beta) + \sin(s-\gamma) + \sin(s-\delta)\right]$$

$$= (a^2-b^2)\cos s + (a^2+b^2)\left[\cos(s-\alpha-\delta) + \cos(s-\beta-\delta) + \cos(s-\gamma-\delta)\right].$$

Solution by R. Lachlan, B.A.; G. Eastwood, M.A.; *and others.*

Let the tangential equation of the parabola be

$$\text{A}\lambda^2 + \text{B}\mu^2 + 2\text{F}\mu\nu + 2\text{G}\nu\lambda + 2\text{H}\lambda\mu = 0;$$

then the equation of the directrix in Cartesian coordinates is (Salmon's *Conics*, Art. 294) $2\text{G}x + 2\text{F}y = \text{A}+\text{B}$. But the tangential equation of the parabola must be of the form

$$[a\lambda\cos\tfrac{1}{2}(\alpha+\beta) + b\mu\sin\tfrac{1}{2}(\alpha+\beta) + \nu\cos\tfrac{1}{2}(\alpha-\beta)]$$

$$\times [a\lambda\cos\tfrac{1}{2}(\gamma+\delta) + b\mu\sin\tfrac{1}{2}(\gamma+\delta) + \nu\cos\tfrac{1}{2}(\gamma-\delta)] + k[a^2\lambda^2 + b^2\mu^2 - \nu^2] = 0.$$

Comparing the two equations, we have

$$k = \cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\gamma-\delta).$$

$$\text{A}+\text{B} = a^2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\gamma+\delta) + b^2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\gamma+\delta)$$
$$+ (a^2+b^2)\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\gamma-\delta),$$

$$2\text{G} = a[\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\gamma-\delta) + \cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\gamma+\delta)],$$

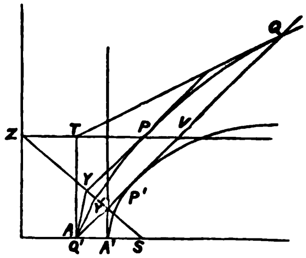
$$2\text{F} = b[\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\gamma-\delta) + \sin\tfrac{1}{2}(\gamma+\delta)\cos\tfrac{1}{2}(\alpha-\beta)].$$

Hence the equation of the directrix is

$$ax[\cos(s-\alpha) + \cos(s-\beta) + \cos(s-\gamma) + \cos(s-\delta)]$$
$$+ by[\sin(s-\alpha) + \sin(s-\beta) + \sin(s-\gamma) + \sin(s-\delta)] + (a^2-b^2)\cos s$$
$$+ (a^2+b^2)[\cos(s-\alpha-\beta) + \cos(s-\alpha-\gamma) + \cos(s-\alpha-\delta)] = 0.$$

7519. (By R. TUCKER, M.A.)—**AP, A′P′ are two confocal and coaxial parabolas, the parameter of the former being twice that of the latter; prove that, if any chord QQ′ be tangential to the inner curve, then the distance of the mid-point of QQ′ from the directrix of AP is trisected where it meets AP, and the tangents at Q, Q′ pass through the other point of trisection.**

Solution by R. LACHLAN, B.A.; R. KNOWLES, B.A., L.C.P.; *and others.*



Let QQ′ be the chord touching A′P′ in P′, and V the mid-point of QQ′; and let the diameter through V meet the curve in P, the tangents at Q, Q′ in T, and the directrix of AP in Z. Then SZ must be perpendicular to QQ′, and the tangent at P; hence, if SZ meet QQ′ in Y′, and the tangent at P in Y, Y, Y′ must lie on the tangents at A, A′ to the parabolas; and evidently since

$$SA = 2SA', \quad YY' = \tfrac{1}{3}SZ,$$

therefore $\quad PV = \tfrac{1}{3}ZV$;

and since PT = PV it follows that ZV is trisected in P and T.

7366. (By C. LEUDESDORF, M.A.)—Two particles A and C, each of mass m', are connected with each other by an elastic string whose modulus of elasticity is λ and whose unstretched length is l; and they are connected with another particle B of mass m by two massless rods, each of length a. The system lies on a smooth horizontal table, and is held so as to form a straight line ABC. The constraints at A and C are removed, and at the same instant each of the particles A and C is projected with velocity v in a direction at right angles to ABC. Find the stress along either rod at the moment when the particles form an equilateral triangle.

Solution by D. EDWARDES; Professor NASH, M.A.; *and others.*

Since evidently m moves in a straight line, if x be its distance from a fixed point at time t, x', y' the coordinates of m', and θ the angle between the rods, then

$$(m+2m')\frac{dx}{dt} - 2am'\sin\theta\frac{d\theta}{dt} = 2m'v \text{ (no external forces).}$$

Again, since $x' = x + a\cos\theta$, $y' = a\sin\theta$, the kinetic energy is

$$\frac{2m'^2v^2 - 2a^2m'^2\sin^2\theta\,\dot{\theta}^2}{m+2m'} + m'a^2\dot{\theta}^2.$$

Subtracting $m'v^2$ from this, and equating the result to the work done by the tension, we get at once

$$\dot{\theta}^2 = \frac{mm'v^2 + 2\lambda(m+2m')\left(\dfrac{a^2}{l}\cos^2\theta - a + a\sin\theta\right)}{m'(m+2m')a^2 - 2a^2m'^2\sin^2\theta}.$$

Differentiating, and putting $\theta = 30°$,

$$\ddot{\theta} = \frac{\lambda\,(m+2m')\left\{a\,(m+\tfrac{1}{4}m') - \frac{a^2}{l}\,m\right\}\sqrt{3} + mm'^2v^2\sqrt{3}}{a^2m'\,(m+\tfrac{3}{2}m')^2}.$$

Now, when $\theta = 30°$, $\qquad \dfrac{m+2m'}{am'}\,\ddot{x} = \sqrt{3}\,\dot{\theta}^2 + \ddot{\theta}$;

whence, substituting, we have, when $\theta = 30°$,

$$\frac{a}{\sqrt{3}}\,\ddot{x} = \frac{2mm'v^2 + \lambda\left\{\frac{a^2}{l}\,(m+\tfrac{3}{4}m') - 2am'\right\}}{(m+\tfrac{3}{2}m')^2}.$$

Let R be the stress along either rod, then $m\dfrac{d^2x}{dt^2} = \mathrm{R}\sqrt{3}$,

therefore
$$\mathrm{R} = m\,\frac{2mm'v^2 + \lambda\left\{\frac{a^2}{l}\,(m+\tfrac{3}{4}m') - 2am'\right\}}{a\,(m+\tfrac{3}{2}m')^2}$$
$$= 2m\,\frac{4mm'lv^2 + a\lambda\left\{2am + (9a-4l)\,m'\right\}}{al\,(2m+3m')^2}.$$

7494. (By W. J. C. SHARP, M.A.)—Show that

$$\int_{-1}^{1} \frac{d^{n-m}\,(x^2-1)^n}{dx^{n-m}} \cdot \frac{d^{n+m}\,(x^2-1)^n}{dx^{n+m}}\,dx = (-1)^m\,\frac{2^{2n+1}}{2n+1}\,(n\,!)^2.$$

Solution by D. EDWARDES; BELLE EASTON; *and others.*

Let $y = (1-x^2)^n$, and $\mathrm{D} \equiv \dfrac{d}{dx}$. Then, integrating by parts,

$$u = \Big[\mathrm{D}^{n+m-1}y\,\mathrm{D}^{n-m}y - \mathrm{D}^{n+m-2}y\,\mathrm{D}^{n-m+1}y + \dots$$
$$\dots(-)^{m-1}\,\mathrm{D}^n y\,\mathrm{D}^{n-1}y\Big]_{-1}^{+1} + (-)^m\int_{-1}^{+1}\mathrm{D}^n y\,\mathrm{D}^n y\,dx.$$

Now, if P_n be the coefficient of z^n in the expansion of $(1+2zx+z^2)^{-\frac{1}{2}}$, by a well-known theorem $\displaystyle\int_{-1}^{+1}\mathrm{P}_n\,\mathrm{P}_n\,dx = \frac{2}{m+1}$, where $\mathrm{P}_n = \dfrac{\mathrm{D}^n\,(1-x^2)^n}{2^n\,\Gamma\,(n+1)}$. Also, from the equation $(1-x^2)\,\dfrac{dy}{dx} + 2nxy = 0$, we find, when $x = 1$ or -1, $\mathrm{D}^r y = 0\,[r<n]$; hence $u = (-)^m\,\dfrac{2^{2n+1}}{2n+1}\,(n\,!)^2$.

7273. (By A. MCMURCHY, B.A.)—Prove that, if radii be drawn to a sphere parallel to the principal normals at every point of a closed curve of continuous curvature, the locus of their extremities divides the sphere into two equal parts.

Solution by the PROPOSER; Professor MATZ, M.A.; *and others.*

Let $d\theta$ be the angle between two consecutive osculating planes, then the angle in the spherical polygon formed by extremities of radii $= \pi - d\theta$, and sum of angles of polygon $= \Sigma(\pi - d\theta)$; hence

Area sought $= R^2[\Sigma(\pi - d\theta) - (n-2)\pi] = 2\pi R^2 - \Sigma d\theta.$

But, in going right round, the curve $d\theta$ is as often positive as negative; therefore $\Sigma d\theta = 0$, therefore area $= 2\pi R^2$, therefore &c.

7460. (By Professor WOLSTENHOLME, M.A., D.Sc.)—

If $$x^n = \cos^n\theta + \sin^n\theta,$$

prove that $$x^{2n-1}\left(\frac{d^2x}{d\theta^2} + x\right) = (n-1)(\sin\theta\cos\theta)^{n-2}.$$

Solution by J. S. JENKINS; R. KNOWLES, B.A.; *and others.*

$$x^{n-1}\frac{dx}{d\theta} = \sin^{n-1}\theta\cos\theta - \cos^{n-1}\theta\sin\theta;$$

and again, differentiating and multiplying each side by x^n, we have

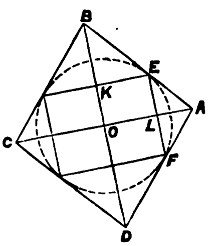
$$(n-1)\,x^{2n-2}\left(\frac{dx}{d\theta}\right)^2 + x^{2n-1}\frac{d^2x}{d\theta^2} = (n-1)(\sin^{n-2}\theta + \cos^{n-2}\theta)\,x^n - nx^{2n};$$

whence, by transposing and reducing, we have

$$x^{2n-1}\left(\frac{d^2x}{d\theta^2} + x\right) = (n-1)(\sin^{n-2}\theta\cos^n\theta + \cos^{n-2}\sin^n\theta) = \text{the result.}$$

7471. (By D. EDWARDES. Suggested by Quest. 7434.)—If an ellipse be inscribed in a rectangle, prove that the perimeter of the quadrilateral formed by joining the points of contact is constant.

Solution by the Rev. T. C. SIMMONS, M.A.; R. KNOWLES, B.A.; *and others.*



Let E be any point of contact on a side AB of the rectangle. Then an ellipse can be drawn through E, as in the figure, to touch the sides symmetrically, and with its centre at O. Also, since five conditions are given, viz., two consecutive points at E and three other tangents, this ellipse is the only one that can be drawn through E; that is, all the ellipses must be symmetrical with respect to the diagonals. Hence it follows at once that the quadrilateral joining the points of contact has its sides parallel to the diagonals; hence its perimeter is constant.

5421. (By Professor Cayley, F.R.S.) — Suppose $S_x = m_1(x-a_1)$, $m_2(x-a_2)$, $m_3(x-a_3)$, $m_4(x-a_4)$; where, for any given value of x, we write $+$, $-$, or 0, according as the linear function is positive, negative, or zero, and where the order of the terms is not attended to. If x is any one of the values a_1, a_2, a_3, a_4, the corresponding S is $0+++$, $0---$, $0++-$, or $0+--$; and if I denote indifferently the first or second form, and R denote indifferently the third or fourth form, then it is to be shown that the four S's are R, R, R, R, or else R, R, I, I.

Solution by W. J. C. Sharp, M.A.

If a_1, a_2, a_3, a_4 be in ascending order of magnitude, then, if the m's be all positive, the S's are I, R, R, I, being

0	+	+	+
−	0	+	+
−	−	0	+
−	−	−	0

and the signs in each column will change sign with the corresponding m. Now a change of sign in either outer column leaves the result R, R, I, I, and one in either or both the middle columns gives R, R, R, R; whilst these changes, in addition to the change of one or both the outer columns, give R, R, I, I.

7537. (By Professor Townsend, F.R.S.)—An ellipsoidal shell being supposed, by a small movement of rotation round an arbitrary axis passing through the centre of its inner surface, to put into irrotational strain a contained mass of incompressible fluid completely filling its interior; investigate, in finite terms, the equations of the displacement line-system of the strain.

Solution by (1) C. Graham, M.A.; (2) *the* Proposer.

1. Let ω_1, ω_2, ω_3, be the component angular velocities about the axis of the ellipsoid, and V the velocity-potential referred to axes which at any instant coincide with the axes of the ellipsoid; then we have

$$\frac{dV}{dx}\frac{x}{a^2}+\frac{dV}{dy}\frac{y}{b^2}+\frac{dV}{dz}\frac{z}{c^2}$$
$$=(\omega_2 z-\omega_3 y)\frac{x}{a^2}+(\omega_3 x-\omega_1 z)\frac{y}{b^2}+(\omega_1 y-\omega_2 x)\frac{z}{c^2}$$

along the surface, and $\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0$ throughout the liquid; whence we easily deduce

$$V=\omega_1\frac{(b^2-c^2)}{b^2+c^2}yz+\omega_2\frac{(c^2-a^2)}{c^2+a^2}xz+\omega_3\frac{(a^2-b^2)}{a^2+b^2}xy;$$

which, by giving different values to V, represents a series of similar and concentric hyperboloids, having their common centre at the centre of the shell. Transform this system to its axes, and let its equation be $\frac{x^2}{\alpha}+\frac{y^2}{\beta}+\frac{z^2}{\gamma}=K$, where K is the parameter; then the directions of the velocity of any particle of fluid are given by the equations

$$\frac{\frac{dx}{x}}{\alpha} = \frac{\frac{dy}{y}}{\beta} = \frac{\frac{dz}{z}}{\gamma}, \text{ or, integrating, } \frac{x^\alpha}{\lambda} = \frac{y^\beta}{\mu} = \frac{z^\gamma}{\nu};$$

where λ, μ, ν are arbitrary, and α, β, γ known constants.

2. Denoting by a, b, c the three semi-axes of the inner surface of the shell, by p, q, r the three components of the rotation with respect to their directions, by x, y, z the coordinates of any point of the fluid with respect to the same, and by ϕ the potential of the strain; then since, as is well-known,

$$\phi = p\,\frac{b^2-c^2}{b^2+c^2}\,yz + q\,\frac{c^2-a^2}{c^2+a^2}\,zx + r\,\frac{a^2-b^2}{a^2+b^2}\,xy = fyz + gzx + hxy,$$

we have, therefore, for the differential equations of the line-system in question, $\frac{dx}{gz+hy} = \frac{dy}{hx+fz} = \frac{dz}{fy+gx}$; the complete integrals of which in finite terms are, as is also well-known,

$$c_1\,(l_1x+m_1y+n_1z)^{\frac{1}{\lambda_1}} = c_2\,(l_2x+m_2y+n_2z)^{\frac{1}{\lambda_2}} = c_3\,(l_3x+m_3y+n_3z)^{\frac{1}{\lambda_3}};$$

where λ_1, λ_2, λ_3 are the roots of the equation $\lambda^3-(f^2+g^2+h^2)\,\lambda-2fgh=0$; $l_1m_1n_1$, $l_2m_2n_2$, $l_3m_3n_3$ the corresponding values of l, m, n as given by the equations $hm+gn-l\lambda=0$, $fn+hl-m\lambda=0$, $gl+fm-n\lambda=0$; and c_1, c_2, c_3 any arbitrary constants the ratios of which are given for each particular line of the system with any single point $x'y'z'$ of its course.

7538. (By Professor HAUGHTON, F.R.S.)—Show that the law of propagation of heat in a solid sphere is $\frac{dv}{dt} = a\left(\frac{d^2v}{dx^2} + \frac{2}{x}\frac{dv}{dx}\right)$.

Solution by C. GRAHAM, M.A.

The general law of propagation of heat in an isotropic solid is

$$\frac{dv}{dt} = a\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right),$$

where a depends on the conductivity and specific heat of the solid. Transforming this to polar coordinates by the well-known transformation, it becomes

$$\frac{dv}{dt} = a\left(\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} + \frac{1}{r^2}\frac{d^2v}{d\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{d^2v}{d\phi^2}\right);$$

but v is independent of θ and ϕ since the solid is a sphere equally hot at points equidistant from the centre, therefore the equation reduces to the form given in the question.

7513. (By Professor MINCHIN, M.A.)—Give a simple geometrical proof of the existence and fundamental property of the Instantaneous-Acceleration Centre in the uniplanar motion of a rigid body.

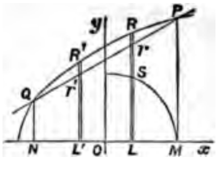
Solution by C. Graham, M.A.

Suppose A and A′ to be two consecutive positions of the centre of instantaneous rotation, and P any point. If the acceleration of P is to be zero, we must have the velocity of P the same in direction and magnitude when the body is rotating about the two points A and A′. Therefore, if ω and ω' be the angular velocities in these two positions, we must have $AP \cdot \omega = A'P \cdot \omega'$ to make the velocities equal. Therefore P must lie on a known circle, since the ratio of AP to A′P is known; and to make the velocities parallel we must have AP parallel to A′P′ when P′ is the second position of P, and therefore $AA' \sin(PA'A) = AP \cdot \phi$ when ϕ is the small angle through which the body has turned in going from the first to its second position. This determines the angle PA′A, and therefore the position of P on the circle already found. So we see there is one position, and only one position, of P.

Again, since the acceleration of P is zero, the acceleration of any point relative to P is its absolute acceleration; but, if Q is any point, its acceleration relative to P along $PQ = PQ \cdot \omega^2$; and perpendicular to $PQ = PQ \cdot \dfrac{d\omega}{dt}$, and therefore the angle which the resultant acceleration makes with PQ is $= \tan^{-1}\left(\dfrac{d\omega}{dt} / \omega^2\right)$, which is independent of the position of Q.

7483. (By Professor Wolstenholme, M.A., Sc.D.)—In Walton's *Mechanical Problems* (3rd ed., p. 19, "Centres of Gravity of Solids of Revolution," Ex. 10) it is stated that the centroid of the solid formed by scooping out a cone from a paraboloid of revolution, the bases and vertices of the two solids being coincident, bisects the axis; prove that (1) this is true for the volume formed by the revolution of any segment, cut off by a chord PQ, from any conic, about an axis of the conic, provided PQ does not cut the axis; also, more generally, (2) if PM, QN be drawn perpendicular to the axis, and a sphere be described on MN as diameter, the centroid of any part of the volume generated by the segment, intercepted between two planes perpendicular to the axis of revolution, is coincident with the centroid of the volume of the sphere intercepted between the same two planes.

Solution by T. Woodcock, B.A.; Professor Matz, M.A.; *and others.*



1. Taking the axis MN for axis of x, and the middle point O of MN for origin, the equation to the conic may be written $y^2 = Ax^2 + Bx + C$. Let $PM = h$, $QN = h'$, $MN = 2a$. Consider two equally narrow strips RL, R′L′, drawn parallel to Oy, on opposite sides of it, and equidistant from it, meeting the curve in R, R′, the chord PQ in r, r', and the axis in L, L′. Let $OL = \lambda = OL'$. We have

$$h^2 = Aa^2 + Ba + C, \quad h'^2 = Aa^2 - Ba + C, \quad RL^2 = A\lambda^2 + B\lambda + C.$$

Also $rL = \dfrac{h-h'}{2a}\lambda + \dfrac{h+h'}{2}$; therefore $RL^2 - rL^2 = \dfrac{C - hh' - Aa^2}{2a^2}(a^2 - \lambda^2)$.

Similarly $RL'^2 - r'L'^2$ is equal to the same quantity. Therefore the volumes generated by the elements Rr and $R'r'$, when the figure revolves round the axis of x, are equal. Therefore the centre of gravity of the volume generated from the segment PQ is at O.

2. If the circle on MN as diameter meet RL in S, $SL^2 = a^2 - \lambda^2$, therefore $RL^2 - rL^2$ varies as SL^2. Therefore the volumes generated by the elements Rr and SL are always in the same proportion. Hence the second part of the theorem follows.

[Otherwise: $\bar{x} = \int x(y_2^2 - y_1^2)\,dx \div \int (y_2^2 - y_1^2)\,dx$, and $y_2^2 - y_1^2$ is a quadratic function of x vanishing at P, Q, when $x = x_1$ or x_2, therefore $\bar{x} = \dfrac{\int x(x-x_1)(x_2-x)\,dx}{\int (x-x_1)(x_2-x)\,dx}$, which is the expression for the centroid of the volume of the sphere on MN as diameter, either in part or whole.]

7413. (By the Rev. T. P. Kirkman, M.A., F.R.S.)—Prove that no polyedron can have a seven-walled frame of pentagons.

Solution by the Proposer.

It should have been added, *if the ray-points of the frame are all triaces;* but it is best to consider only solids whose summits are all triaces, and whose least faces are pentagons. We have first to reproduce a theorem of Euler's. Let a solid having only triad summits have a_3 triangles, a_4 quadrilaterals, a_5 5-gons, ... a_m m-gons ... If from the four equations $e = s + f - 2$, $f = a_3 + a_4 + a_5 + \&c.$, $2e = 3s = 3a_3 + 4a_4 + 5a_5 + \&c.$, we eliminate e, s, and f, a_6 disappears, and we get Euler's result,

$$a_5 = 12 + a_7 + 2(a_8 - a_4) + 3(a_9 - a_3) + 4a_{10} + \ldots + ma_{6+m} + \ldots \ (m > 4),$$

which, when $a_3 = a_4 = 0$, becomes $a_5 = 12 + \mathrm{S}ma_{6+m}$.

On all the edges of a $(6+r)$-gon A draw 6-gons, making $6+r$ summits 665, and $12+2r$ summits 655 with a circle of $12+2r$ pentagons, within which draw $6+r$ more 6-gons collateral with a central $(6+r)$-gon A′, and making with the same 5-gons the like summits. We thus get a $(26+4r)$-edron P, which has no frame, but a circle $C_{(12+2r)} = 1$.

If a $(6+r)$-walled frame F is possible, F can be imposed on the $(6+r)$-gon A′, and equally well on any $(6+r)$-gonal face of any polyedron. Let F be imposed on A′, a face of P. The frame F will have a contour $c_1, c_2, \ldots c_{5+r}, c_{6+r}$, $(c_m \overline{\gtrless} 0)$. The 6-gonal wall which carries c_m ray-points becomes a $(6+c_m)$-gon, which will contribute by Euler's theorem c_m pentagons to the a_5 of the completed solid: that is, if $c_1 + c_2 + \ldots = \mathrm{R}$, the $(6+r)$ walls will contribute to that a_5 R pentagons. The $(6+r)$-gon A, the only non-pentagon > 6 besides these walls will contribute to that a_5 r pentagons, and the completed solid can have no more than $a_5 = 12 + r + \mathrm{R}$ pentagons. In imposing F we have made no change in the $12+2r$ pentagons of P, and we have added to them not fewer than R more.

It follows that $12 + 2r + \mathrm{R} \not> 12 + r + \mathrm{R}$; Q.E.A., if $r > 0$.

7535. (By R. LACHLAN, B.A.)—Prove that, if $\alpha < \frac{1}{2}\pi$, and n be positive and < 1, $$\int_0^\infty \frac{x^n dx}{1 + 2x\cos\alpha + x^2} = \frac{\pi}{\sin n\pi}\,\frac{\sin n\alpha}{\sin\alpha},$$
and $$\int_0^\infty \frac{x^{n-1}dx}{1 + 2x\cos\alpha + x^2} = \frac{\pi}{\sin n\pi}\,\frac{\sin(1-n)\alpha}{\sin\alpha}.$$

Solution by Professor WOLSTENHOLME, Sc.D.; R. KNOWLES, B.A.; *and others.*

By a well-known formula, $\int_0^\infty \frac{x^{n-1}dx}{x+a} = \frac{\pi}{\sin n\pi}\,a^{n-1}$, if $n > 0 < 1$; for all values of a, real or impossible. Putting $a = \cos\alpha + i\sin\alpha$, and therefore $\frac{1}{x+a} = \frac{x + \cos\alpha - i\sin\alpha}{x^2 + 2x\cos\alpha + 1}$, and denoting the integrals proposed by U, V, respectively, we have
$$U + (\cos\alpha - i\sin\alpha)\,V = \frac{\pi}{\sin n\pi}\left[\cos(n-1)\alpha + i\sin(n-1)\alpha\right];$$
whence $U + V\cos\alpha = \frac{\pi}{\sin n\pi}\cos(1-n)\alpha$, $V\sin\alpha = \frac{\pi}{\sin n\pi}\sin(1-n)\alpha$;
whence the results stated.

In these results, α is an angle determined as $\cos^{-1}(\cos\alpha)$, and the limits are accordingly 0 and π; but, since writing $-\alpha$ for α does not alter either member, the results will be true from $\alpha = -\pi$ to $\alpha = \pi$.

Both results are included in
$$\int_0^\infty \frac{x^{n-1}}{x^2 + 2x\cos\alpha + 1}\,dx = \frac{\pi}{\sin n\pi}\,\frac{\sin(1-n)\alpha}{\sin\alpha},$$
if we take $$n > 0 < 2, \quad \alpha > -\pi < \pi.$$

Writing x^p for x, $\frac{n}{p}$ for n, and $p\alpha$ for α, we get
$$\int_0^\infty \frac{x^{n-1}dx}{x^{2p} + 2x^p\cos p\alpha + 1} = \frac{\pi}{p\sin\frac{n\pi}{p}}\,\frac{\sin(p-n)\alpha}{\sin p\alpha},$$
where p is positive, $n > 0 < 2p$, and $p\alpha > -\pi < \pi$; and in this form, which is not really more general than the one proposed, the result will be found in WOLSTENHOLME's *Math. Problems* [1919 (50)].

If we put $\alpha = -\pi$ or π (in U or V), both members become ∞, but probably the limiting ratio of the two members as α tends to π is *not* one of equality. [See De MORGAN's *Calculus*, p. 666.]

7439. (By R. RAWSON.)—Two inclined planes of the same height and inclination α, β, are placed back to back, with an interval between them ($2a$). Two weights P, Q are placed one on each inclined plane, and kept at rest by the connection of an inextensible string, indefinitely long, passing over two small tacks, one at the top of each inclined plane. A weight w, having a vertical velocity (c), is then placed on the string by a smooth ring at a point midway between the inclined planes. Show that the system thereby put in motion will come to rest at a point determined by a root of the quadratic

$$(4P^2 \sin^2 \alpha - w^2) s^2 - \frac{w}{g}(4ga P \sin \alpha + wc^2) s - \left(2Pa \sin \alpha + \frac{wc^2}{4g}\right)\frac{wc^2}{g} = 0.$$

Solution by D. EDWARDES; Professor MATZ, M.A.; *and others.*

Let a fixed horizontal plane below the system be taken as plane of reference, referred to which, let V be the original potential energy of the system P, Q, and h the common altitude of the wedges. Then the whole energy at first is $V + wh + \frac{1}{2}\frac{w}{g}c^2$. Now, since the *ring* is smooth, the tension is the same throughout, and w descends vertically. Let x be the distance it describes before reaching its position of instantaneous rest. Then the sum of the distances described by P and Q along the wedges is $2[(x^2+a^2)^{\frac{1}{2}} - a]$. There is evidently no impulse on the weights, and therefore no energy lost. Also, since the weights P and Q are in equilibrium at first, $P \sin \alpha = Q \sin \beta$. Hence the increase of potential energy of P and Q is $2P \sin \alpha\,[(x^2+a^2)^{\frac{1}{2}} - a]$. Therefore

$$V + wh + \tfrac{1}{2}\frac{w}{g}c^2 = \text{whole potential energy at last}$$
$$= w(h-x) + V + 2P \sin \alpha\,[(x^2+a^2)^{\frac{1}{2}} - a],$$

or
$$\tfrac{1}{2}\frac{w}{g}c^2 + wx = 2P \sin \alpha\,[(x^2+a^2)^{\frac{1}{2}} - a],$$

which, when rationalized, gives the required result.

7396. (By D. EDWARDES.)—Prove that

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} F(1 - \sin\theta\cos\phi)\sin\theta\, d\theta\, d\phi = \tfrac{1}{2}\pi\int_0^1 F(u)\, du.$$

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.

The limits of integration show that the integral is extended to the surface of a quadrantal triangle traced on a sphere of radius unity, or, taking as axes of x, y, z the radii of the sphere drawn to the angular points of the triangle, and supposing the radius vector to any variable point on the surface to make with these axes angles α, β, γ, we have $\alpha = \theta$, $\cos\beta = \sin\theta\cos\phi$, $\cos\gamma = \sin\theta\sin\phi$; therefore, if $d\Omega$ denote the element of surface,

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} F(1-\sin\theta\cos\phi)\sin\theta\, d\theta\, d\phi = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} F(1-\cos\beta)\, d\Omega$$

$$= \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} F(1-\cos\beta)\sin\beta\, d\beta\, d\chi = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} F(2\sin^2\tfrac{1}{2}\beta)\, 2\sin\tfrac{1}{2}\beta\cos\tfrac{1}{2}\beta\, d\beta\, d\chi,$$

or, putting $2\sin^2\frac{1}{2}\beta = u$, $\quad = \int_0^1\int_0^{\frac{1}{2}\pi} F(u)\, du\, d\chi = \frac{1}{2}\pi\int_0^1 F(u)\, du.$

7399. (By ASÛTOSH MUKHOPÂDHYÂY.)—A sphere is described round the vertex of a cone as centre; prove that the latus rectum of any section of the cone, made by any variable tangent plane to the sphere, is equal to the diameter of the sphere, multiplied by the tangent of the semi-vertical angle of the cone.

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.

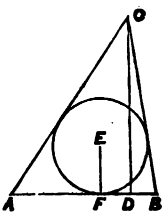
It is geometrically evident that, if two given surfaces of the second degree touch along a conic, any plane cutting them both will cut the two surfaces in two conics which have double contact along the line in which this plane cuts the plane of contact of the two given surfaces; if, therefore, the plane *touch* one of the surfaces at one of its umbilici, it will cut the other surface in a conic with which this point of contact, an evanescent circle, will have (imaginary) double contact along the line in which this tangent plane cuts the plane of contact of the two given surfaces, or the point of contact is a focus of the conic and the line a directrix. It is therefore evident that the foci of any plane section of a right cone of revolution will be its points of contact with the two spheres which can be described touching it and touching the cone along a circle whose plane is perpendicular to the axis of the cone. Let then from C, the vertex of the cone, the perpendicular CD (p) be drawn to the plane of the conic, and let a plane be drawn through CD and EF (the line drawn from the centre of either of the spheres described as above indicated, to the point in which it touches the plane section), cutting the cone in the lines CA, CB; with the usual notation we have, in the triangle ACB,

$$\sin^2 \tfrac{1}{2}C = \frac{(s-a)(s-b)}{ab} = \frac{(s-a)(s-b)\sin C}{p \cdot c};$$

hence we have

$$\frac{\text{Parameter}}{4} = \frac{\text{AF} \cdot \text{FB}}{\text{AB}} = \frac{(s-a)(s-b)}{c} = \tfrac{1}{4} p \tan \tfrac{1}{2}C;$$

therefore Parameter $= 2p \tan \tfrac{1}{2}C$, but $2p$ is the diameter of the sphere referred to in the question.



7363. (By G. G. MORRICE, B.A.)—If $| A_1, B_2, C_3 |$ be the reciprocal determinant of $| a_1, b_2, c_3 |$, prove that

$$(1)\ \Sigma (a_1^2 + a_2^2 + a_3^2)(A_1^2 + A_2^2 + A_3^2) = 3 \,| a_1, b_2, c_3 |^2 + 2\Sigma (a_1^2 + a_2^2 + a_3^2)(b_1c_1 + b_2c_2 + b_3c_3)^2 - 6 (b_1c_1 + b_2c_2 + b_3c_3)(c_1a_1 + c_2a_2 + c_3a_3)(a_1b_1 + a_2b_2 + a_3b_3).$$

$$(2)\ a_1 (A_2 - A_3) + a_2 (A_3 - A_1) + a_3 (A_1 - A_2) = (b_1 + b_2 + b_3)(c_1a_1 + c_2a_2 + c_3a_3) - (c_1 + c_2 + c_3)(a_1b_1 + a_2b_2 + a_3b_3).$$

Solution by the PROPOSER.

$$(A_1^2 + A_2^2 + A_3^2)(a_1^2 + a_2^2 + a_3^2) - (A_1a_1 + A_2a_2 + A_3a_3)^2 = (a_2A_3 - a_3A_2)^2 + (a_3A_1 - a_1A_3)^2 + (a_1A_2 - a_2A_1)^2.$$

But $\qquad a_2A_3 - a_3A_2 = a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)$
$= b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3) = b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3).$

Adding the two similar expressions for $(a_3A_1 - a_1A_3)$ and $(a_1A_2 - a_2A_1)$, we get (2), and by squaring we get (1).

7475. (By J. O'Regan.)—The figures 142857 are arranged at random as the period of a circulating decimal, which is then reduced to a vulgar fraction in lowest terms; show that the odds are 119 : 1 against the denominator being 7.

Solution by A. Martin, B.A.; Rev. T. C. Simmons, M.A.; *and others.*

There are 6 ways in which the denominator can be 7, viz., when the decimal is $\cdot\dot{1}4285\dot{7}$, $\cdot\dot{4}2857\dot{1}$, $\cdot\dot{2}8571\dot{4}$, $\cdot\dot{8}5714\dot{2}$, $\cdot\dot{5}7142\dot{8}$, $\cdot\dot{7}1428\dot{5}$. In all other cases the denominator is not 7. But there are 720 − 6, or 714, of these cases; hence the required odds are 714 : 6, or 119 : 1.

7236. (By the Rev. T. W. Openshaw, M.A.)—On AB, a chord of an ellipse, as diameter, a circle is drawn intersecting the ellipse again in C, D; if AB, CD are parallel to a pair of conjugate diameters: show that the locus of their intersection is $b^3x + a^3y = 0$.

Solution by the Proposer.

AB and CD must be parallel to equi-conjugate diameters, therefore their equations are of form $bx + ay + k = 0$, $bx - ay + k' = 0$; equation to conic through A, B, C, D is $(bx + ay + k)(bx - ay + k') = l(a^2y^2 + b^2x^2 - a^2b^2)$, the condition that this is a circle gives $l = \dfrac{b^2 + a^2}{b^2 - a^2}$; the coordinates of the centre are $\dfrac{b(k + k')(b^2 - a^2)}{4a^2b^2}$, $\dfrac{a(k' - k)(b^2 - a^2)}{4a^2b^2}$; if this is on $bx + ay + k = 0$, we get $(k' + k)b^2 = (k' - k)a^2$; whence, eliminating k, k', the locus is as stated.

7522. (By W. J. C. Sharp, M.A.)—Prove that (1) any two conics are polar reciprocals with respect to a third; (2) the same triangle is self-reciprocal with respect to all three, and the equation of the auxiliary conic, referred to this, may be derived from those to the other two by taking each coefficient proportional to the geometrical mean between the corresponding coefficients of the reciprocal conics; (3) the analogous proposition is true of quadrics.

Solution by Professor WOLSTENHOLME, M.A., Sc.D.

Referring the two conics to their common self-conjugate triangle, we may suppose their equations to be

$$x^2+y^2+z^2 = 0,\ (px)^2+(qy)^2+(rz)^2 = 0 \ \ldots\ldots\ldots\ldots(1,\ 2),$$

and, if we take another conic $lx^2+my^2+nz^2=0$ ....(3), with respect to which the triangle is self-conjugate, the reciprocal polars of (1), (2) with respect to (3) will be respectively $l^2x^2+m^2y^2+n^2z^2=0$, $\frac{l^2x^2}{p^2}+\frac{m^2y^2}{q^2}+\frac{n^2z^2}{r^2}=0$. Hence, if $\frac{l^2}{p^2}=\frac{m^2}{q^2}=\frac{n^2}{r^2}$, either of the conics (1), (2) is the reciprocal polar of the other with respect to (3). Thus there are *four* such auxiliary conics (a well-known result). Obviously, we shall have exactly similar equations for conicoids, and there will be eight auxiliary conicoids with respect to which either of two given conicoids is the reciprocal polar of the other. [The conics ϕ and F are polar reciprocals with respect to the same conics as U and V. And in space τ and T and τ' and T′ are polar reciprocals with respect to the same quadrics as U and V.]

7530. (By R. KNOWLES, B.A., L.C.P.)—From a point A a perpendicular AD is drawn to a straight line BC given in position, and the inscribed circle of the triangle ABC passes through the orthocentre; prove that the maximum value of its radius is one-half of AD.

Solution by the Rev. T. C. SIMMONS, M.A.; R. LACHLAN, B.A.; *and others.*

Let O be the orthocentre, and I the incentre; then $IO^2 = 2r^2 - AO.OD$ (*Reprint*, Vol. 39, p. 99); therefore, if $IO = r$, $r^2 = AO.OD$; and, since A and D are given, the maximum value of r is half of AD.

7458. (By Professor WOLSTENHOLME, M.A., D.Sc.)—If n, r be positive integers, and
$$x^r y = \sin x,\quad x^{n+r}\frac{d^n y}{dx^n} = z,$$
prove that, according as $n+r$ is even or odd,
$$\frac{d^r z}{dx^r} = (-1)^{\frac{n+r}{2}} x^n \sin x,\quad \text{or}\quad (-1)^{\frac{n+r-1}{2}} x^n \cos x.$$
$$\left[\text{The results may be written } \frac{d^r z}{dx^r} = x^n \sin\left\{(n+r)\frac{\pi}{2}+x\right\}.\right]$$

Solution by J. HAMMOND, M.A.; G. B. MATHEWS, B.A.; *and others.*

If $D = \frac{d}{dx}$, we have, by the conditions of the question,

$$x^n D^{n+r}(x^r y) = x^n \sin\left[(n+r)\tfrac{1}{2}\pi + x\right],\quad D^r(x^{n+r}D^n y) = D^r z \ldots(1,\ 2),$$

and it only remains to prove that $x^n D^{n+r}(x^r y) = D^r(x^{n+r}D^n y)$.

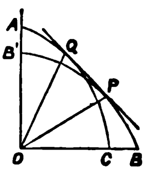
But, by LEIBNITZ's theorem, we have

$$D^r(x^{n+r}D^n y) = x^{n+r}D^{n+r}y + (n+r)\,r x^{n+r-1}D^{n+r-1}y$$
$$+ (n+r)(n+r-1)\frac{r(r-1)}{1\,.\,2}x^{n+r-2}D^{n+r-2}y + \dots = x^n D^{n+r}(x^r y).$$

7155. (By T. WOODCOCK, B.A.)—If P, Q be the points in which the plane through the optic and ray axes intersects the circle of contact PQ of a tangent plane perpendicular to an optic axis of the wave surface of a biaxal crystal, and if a, c, the greatest and least axes of elasticity, be given; prove that, O being the centre of the wave surface, (1) the triangle POQ, (2) the circle of contact PQ, (3) the angle POQ will have their greatest values respectively, when the square of the mean axis b is (i.) the arithmetic, (ii.) the geometric, (iii.) the harmonic mean of a^2 and b^2; and the cone whose vertex is O and base the circle PQ will have its maximum volume when $b^2 = \frac{1}{6}\left[a^2+c^2+(a^4+14a^2c^2+c^4)^{\frac{1}{2}}\right]$.

Solution by the PROPOSER.

In the accompanying figure let BPB′ be a circle of radius b, and AQC an ellipse with axes $2a$, $2c$; and let D = semi-conjugate of OQ; then we have



$$PQ^2 = OQ^2 - OP^2 = a^2+c^2-D^2-b^2 = a^2+c^2-\frac{a^2c^2}{b^2}-b^2,$$
a maximum when $b^2 = ac$,
$$b^2\,.\,PQ^2 = (a^2+c^2)\,b^2 - a^2c^2 - b^4,$$
a maximum when $2b^2 = a^2+c^2$,
$$\tan^2 POQ = \frac{a^2+c^2}{b^2} - \frac{a^2c^2}{b^4} - 1,\ \text{a maximum when } b^2(a^2+c^2) = 2a^2c^2.$$

The quadratic in b^2, got by making the cone's volume a maximum, is
$$b^2(a^2+c^2) + a^2c^2 - 3b^4 = 0,\ \text{therefore, \&c.}$$

7431. (By Professor WOLSTENHOLME, M.A., D.Sc.)—If $2s \equiv \alpha+\beta+\gamma+\delta$, prove that

$$\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\alpha-\delta)\left[\sin(s-\beta)+\sin(s-\gamma)-\sin(s-\alpha)-\sin(s-\delta)\right]^2$$
$$+\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\beta-\delta)\left[\sin(s-\gamma)+\sin(s-\alpha)-\sin(s-\beta)-\sin(s-\delta)\right]^2$$
$$+\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\gamma-\delta)\left[\sin(s-\alpha)+\sin(s-\beta)-\sin(s-\gamma)-\sin(s-\delta)\right]^2$$
$$\equiv -16\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\alpha-\delta)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\beta-\delta)\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\gamma-\delta).$$

Solution by G. HEPPEL, M.A.; G. B. MATHEWS, B.A.; *and others.*

In a rough way this may be said to be shown by observing that the left-hand member vanishes when $\beta=\gamma$, and when $\alpha=\delta$. It is, however, a little troublesome to find the constant multiplier. A fuller and more satisfactory proof is obtained by putting

$$\sin\tfrac{1}{2}(\beta-\gamma)=u,\ \cos\tfrac{1}{2}(\beta+\gamma)=u',\ \sin\tfrac{1}{2}(\alpha-\delta)=p,\ \cos\tfrac{1}{2}(\alpha+\delta)=p',$$
$$\sin\tfrac{1}{2}(\gamma-\alpha)=v,\ \cos\tfrac{1}{2}(\gamma+\alpha)=v',\ \sin\tfrac{1}{2}(\beta-\delta)=q,\ \cos\tfrac{1}{2}(\beta+\delta)=q',$$
$$\sin\tfrac{1}{2}(\alpha-\beta)=w,\ \cos\tfrac{1}{2}(\alpha+\beta)=w',\ \sin\tfrac{1}{2}(\gamma-\delta)=r,\ \cos\tfrac{1}{2}(\gamma+\delta)=r';$$

then, if U = left-hand member, we have

$$U = 4pu(qv'+q'v)(rw'-r'w)+4qv(rw'+r'w)(pu'-p'u)$$
$$+4rw(pu'+p'u)(qv'-q'v),$$
$$\tfrac{1}{4}U = rw'[pq(uv'+u'v)+uv(pq'-p'q)]-r'w[qu(pv'+p'v)-pv(qu'-q'u)]$$
$$+rw(pu'+p'u)(qv'-q'v).$$

Now the following identities hold good :—

$$uv'+u'v=-w\cos\gamma,\quad pq'-p'q=w\cos\delta,$$
$$pv'+p'v=r\cos\alpha,\quad qu'-q'u=r\cos\beta;$$

therefore
$$\frac{U}{4wr}=-pqw'\cos\gamma+uvw'\cos\delta-qur\cos\alpha+pvr'\cos\beta$$
$$+(pu'+p'u)(qv'-q'v)$$
$$=pq[u'v'-w'\cos\gamma]+uv[w'\cos\delta-p'q']$$
$$+qu[p'v'-r'\cos\alpha]-pv[q'u'-r'\cos\beta],$$

and every one of these four terms $=-pquv$, therefore $U=-16pqruvw$.

7343. (By BELLE EASTON.)—If a debating society has to choose one out of five subjects proposed, and 30 members vote each for one subject, show that (1) the votes can fall in 5^{30} ways, and (2) the chance that upwards of twenty votes fall to some one subject will be 5^{-20}.

Solution by W. W. TAYLOR, M.A.; SARAH MARKS; *and others.*

1. Let v, w, x, y, z represent the subjects, then the possible combinations of votes and the relative probability of each combination will be represented by the coefficients in the expansion $(v+w+x+y+z)^{30}$; and the sum of these coefficients is $(1+1+1+1+1)^{30}=5^{30}$.

2. If upwards of 20 fall to one subject, 21 is the least number of votes that can be recorded for that subject; the remainder of the votes is 9. These can be given in 5^9 different ways, and the one subject can be chosen in five different ways; so, the conditions of this part of the problem can be satisfied in 5^{10} different ways, and therefore the chance required is 5^{-20}.

6907. (By S. TEBAY, B.A.)—If A, B, C can do similar pieces of work in a, b, c hours respectively, $(a<b<c)$; and they begin simultaneously, and regulate their labour by mutual interchanges at certain intervals, so that the three pieces of work are finished at the same time: find the number of solutions.

Solution by the PROPOSER; SARAH MARKS; *and others.*

The parts done by each in x hours are $\frac{x}{a}$, $\frac{x}{b}$, $\frac{x}{c}$. Suppose now that B and C change works, while A continues. In y hours more they have done $\frac{y}{a}$, $\frac{y}{a}$, $\frac{y}{c}$. Therefore the work remaining to be done by each is

$$1-\frac{x}{a}-\frac{y}{a},\quad 1-\frac{x}{b}-\frac{y}{c},\quad 1-\frac{x}{c}-\frac{y}{b}.$$

There are two ways of completing the work according to the question, which may be set down as follows:—

A finishes B's or A finishes C's,
B ,, A's ,, C ,, A's,
C ,, C's ,, B ,, B's.

Thus, if the times be equal, we have

$$b\left(1-\frac{x}{a}-\frac{y}{a}\right)=a\left(1-\frac{x}{b}-\frac{y}{c}\right)=c\left(1-\frac{x}{c}-\frac{y}{b}\right),$$

$$c\left(1-\frac{x}{a}-\frac{y}{a}\right)=b\left(1-\frac{x}{b}-\frac{y}{c}\right)=a\left(1-\frac{x}{c}-\frac{y}{b}\right).$$

Putting $s=bc+ca+ab$, $t=a+b+c$, these equations give

$$x=\frac{s^2-3tabc}{s(a-b)(c-b)}b,\quad y=\frac{s-3ab}{s(c-b)}bc \quad\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

$$x=\frac{s^2-3tabc}{s(c-a)(c-b)}c,\quad y=\frac{3ac-s}{s(c-b)}bc \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

In the same way we have the following combinations. Reasoning as above, let C and A change works, while B continues; then let

B finish A's or B finish C's,
A ,, B's ,, C ,, B's,
C ,, C's ,, A ,, A's.

These arrangements give

$$x=\frac{s^2-3tabc}{s(b-a)(c-a)}a,\quad y=\frac{s-3ab}{s(c-a)}ac \quad\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$x=\frac{s^2-3tabc}{s(c-a)(c-b)}c,\quad y=\frac{3bc-s}{s(c-a)}ac \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Again, let A and B change works, while C continues; then let

A finish C's or B finish C's,
C ,, A's ,, C ,, B's,
B ,, B's ,, A ,, A's.

These arrangements give

$$x=\frac{s^2-3tabc}{s(b-a)(c-a)}a,\quad y=\frac{s-3ac}{s(b-a)}ab \quad\ldots\ldots\ldots\ldots\ldots(5),$$

$$x=\frac{s^2-3tabc}{s(a-b)}b,\qquad y=\frac{s-3bc}{s(a-b)}ab\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Now $s^2-3tabc$ is always positive; and, since $a<b<c$, it will be seen that (1) and (6) are inadmissible. It is also evident that (2) and (5) cannot be

both relevant. The whole work is done in $3s^{-1}abc$ hours. If x and y be both positive, we must have $x+y<3s^{-1}abc$. Applying this test to (2), (3), (4), (5), we find

$$2bc > a(b+c),\quad 2ab < c(a+b),\quad 2ac < b(a+c),\quad 2ac > b(a+b)\ldots(2', 5', 3', 4').$$

Here (2′) and (5′) are both possible, but only one is applicable. (3′) and (4′) are inconsistent, one only being applicable. There are therefore only two possible solutions.

If $b < \dfrac{2ac}{a+c}$, (2) and (4) are applicable; if $b > \dfrac{2ac}{a+c}$, (3) and (5) are applicable; if $b = \dfrac{2ac}{a+c}$, (2) and (5), and also (3) and (4), are identical, the work being completed in b hours.

Examples.—The harmonic mean between 4 and 12 is 6; so that, if $a=4$, $b=6$, $c=12$, the whole work is finished in 6 hours; A and C changing works at the end of 3 hours.

Take $a=4$, $b=5$, $c=12$; then

$x = 2\frac{13}{112}$ hours, $y = 1\frac{1}{14}$ hours; $x = 2\frac{13}{112}$ hours, $y = 2\frac{7}{16}$ hours...(2, 4),
the whole work being completed in $5\frac{5}{8}$ hours.

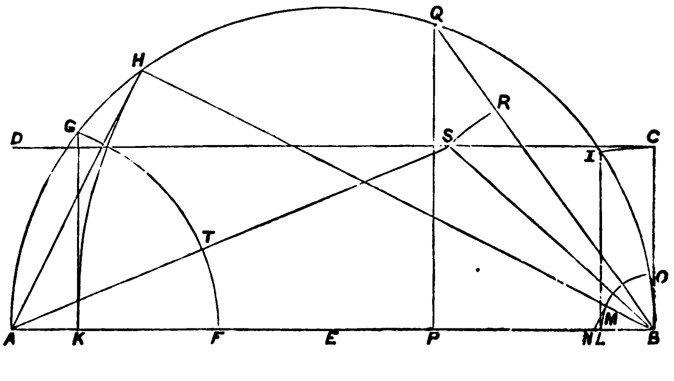
Take $a=4$, $b=7$, $c=12$; then

$x = 2\frac{31}{80}$ hours, $y = 2\frac{17}{20}$ hours; $x = 2\frac{31}{80}$ hours, $y = \frac{14}{15}$ hour...(3, 5),
the whole work being completed in $6\frac{3}{10}$ hours.

7552. (By the Editor.)—In a road parallel to a range, find, by elementary geometry, a point at which the sounds of the firing and of the hit of the bullet would be heard simultaneously.

Solution by (1) D. Biddle; (2) A. H. Curtis, LL.D., D.Sc.

1. Let A be the firing-point, B the target, CD the road, and AF the distance the sound will travel before the bullet reaches B. Draw BC at



right angles to AB; on AB the semi-circle AGHQB; CI with radius BC; IL perpendicular to AB; FG with radius AF; and GK perpendicular to AB. Draw LMO with radius BL, and KH with radius BK. Join HB cutting LMO in M, and draw MN parallel to HA, that is, at right angles to HB. Make NP = $\frac{1}{2}$AB, and draw PQ at right angles to AB, cutting the semicircle in Q. Join BQ, and make QR = $\frac{1}{2}$AF. Finally, with radius BR, draw RS, cutting CD in S. Then AS = BS + AF, and S is the required point.

For it will readily be seen that by construction, and the properties of circles and right-angled triangles, if AB = 1, then $AK = AF^2$, $BL = BC^2$, and $BP = BQ^2$. Now, BR = BS, and BQ = BS + $\frac{1}{2}$AF; therefore $BP = (BS + \frac{1}{2}AF)^2$. But $BP = BN + \frac{1}{4}AB^2$; hence we have

$$BN = (BS + \tfrac{1}{2}AF)^2 - \tfrac{1}{4}AB^2.$$

But $BM = BL = BC^2$ and $BN : BM = AB : BH = AB^2 : AB^2 - AF^2$,

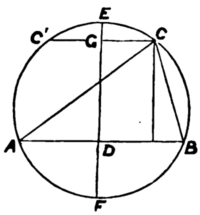
therefore $$(BS + \tfrac{1}{2}AF)^2 - \tfrac{1}{4}AB^2 : BC^2 = AB^2 : AB^2 - AF^2,$$

whence $$BS = \left(\tfrac{1}{4}AB^2 + \frac{AB^2 BC^2}{AB^2 - AF^2}\right)^{\frac{1}{2}} - \tfrac{1}{2}AF,$$

which is the identical value arrived at from the equation

$$(BS^2 - BC^2)^{\frac{1}{2}} + [(BS + AF)^2 \quad BC^2]^{\frac{1}{2}} = AB.$$

Consequently, since it is plain that $(BS^2 - BC^2)^{\frac{1}{2}} + (AS^2 - BC^2)^{\frac{1}{2}} = AB$, it is equally evident that AS = BS + AF, and S is the required position.



2. The question is immediately reducible to the following:—Given, of a plane triangle, altitude, base, and difference of the other two sides, to construct it. Let ACB be the triangle, FE the diameter of its circumscribed circle, which is perpendicular to the base AB, and CG parallel to AB, then EG.DF + GD.DF, = ED.DF = AD^2 = $\frac{1}{4}AB^2$, is known, and EG.DF, = $\frac{1}{4}(AC - CB)^2$, is known; therefore GD.DF is known, and GD, = altitude, is known; therefore DF is known, and therefore GE. If therefore at D, the middle point of AB, we draw a perpendicular and measure off DF, DG, GE, on FE as diameter describe a circle, and through G draw GC parallel to AB; GC will intersect the circle in the required point C, and in another point C′ which is excluded, as, if B be the target, BC is the smaller side. [Question 7265 is related to this problem.]

7416. (By R. Rawson.)—In the Royal Society's *Transactions* (Part III., 1881, pp. 766, 767), Mr. J. W. L. Glaisher has shown, by the assumption of $\Sigma A_r x^{m+r}$ for all positive integral values of r, that (AU + BV) is the general integral of $\frac{d^2\omega}{dx^2} - a^2\omega = \frac{p(p+1)}{x^2}\omega$, where

$$U = x^{-p}\left\{1 - \frac{1}{p-\frac{1}{2}}\frac{a^2x^2}{2^2} + \frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})}\frac{a^4x^4}{2^4.2!} - \&c.\right\},$$

$$V = x^{p+1}\left\{1 + \frac{1}{p+\frac{3}{2}}\frac{a^2x^2}{2^2} + \frac{1}{(p+\frac{3}{2})(p+\frac{5}{2})}\frac{a^4x^4}{2^4.2!} + \&c.\right\}.$$

Show that the restriction imposed upon r is unnecessary, and that, if $m = n-2p$, the general integral of the above differential equation is

$$\omega = A_0x^{n-p}\left\{1 + \frac{a^2x^2}{(n+2)(m+1)} + \frac{a^4x^4}{(n+4)(n+2)(m+3)(m+1)} + \&c.\right\}$$

$$+\frac{n.m-1.A_0}{a^2}x^{n-p-2}\left\{1 + \frac{(n-2)(m-3)}{a^2x^2} + \frac{(n-4)(n-2)(m-5)(m-3)}{a^4x^4} + \&c.\right\}$$

Solution by the Proposer.

Let $\omega = \phi(x) = \Sigma A_r x^{2r+\beta}$

$= x^\beta[A_0 + A_1x^2 + A_2x^4 + \&c.] + x^{\beta-2}[A_{-1} + A_{-2}x^{-2} + A_{-3}x^{-4} + \&c.] \ldots (1).$

Differentiate (1), then

$$\frac{du}{dx} = \Sigma(2r+\beta)A_rx^{2r+\beta-1}, \quad \frac{d^2\omega}{dx^2} = \Sigma(2r+\beta)(2r+\beta-1)A_rx^{2r+\beta-2},$$

$$-\frac{p(p+1)}{x^2}\omega = \Sigma - p(p+1)A_rx^{2r+\beta-2}, \quad -a^2\omega = \Sigma - a^2A_rx^{2r+\beta};$$

hence $\frac{d^2\omega}{dx^2} - a^2\omega = \frac{p(p+1)}{x^2}\omega \ldots\ldots (2),$

if $(2r+\beta+p)(2r+\beta-p-1)A_r = a^2A_{r-1} \ldots\ldots (3),$

the summation extending to all positive and negative integral values of r.

The general integral of (2) is, therefore, $\phi(x)$, which must be of such a form that the coefficients of $x^{2r+\beta-2}$ and $x^{2r+\beta}$ shall satisfy (3).

Put $n = \beta+p$, and $m = \beta-p = n-2p \ldots\ldots (4).$

Equation (3) may now be written

$$(n+2r)(m+2r-1)A_r = a^2A_{r-1} \ldots\ldots (5).$$

From (1) the following values of A_1, A_2, &c., A_{-1}, A_{-2}, &c., may be readily obtained:—

$$A_1 = \frac{a^2A_0}{n+2.m+1}, \quad A_2 = \frac{a^2A_1}{n+4.m+3} = \frac{a^4A_0}{n+4.n+2.m+3.m+1},$$

$$A_3 = \frac{a^2A_2}{n+6.m+5} = \frac{a^6A_0}{n+6.n+4.n+2.m+5.m+3.m+1}, \&c.;$$

$$A_{-1} = \frac{n(m-1)A_0}{a^2}, \quad A_{-2} = \frac{n-2.m-3.A_{-1}}{a^2} = \frac{n-2.n.m-3.m-1.A_0}{a^4},$$

$$A_{-3} = \frac{n-4.m-5.A_{-2}}{a^2} = \frac{n-4.n-2.n.m-5.m-3.m-1.A_0}{a^6}, \&c.$$

Substitute these values in (1), then (ω) is the value assigned in the question. By making $n = 0$, and $m = 1$, successively, there results the two particular solutions obtained by Mr. Glaisher.

7393. (By W. J. McClelland, B.A.)—If from any two points inverse to each other with respect to a given circle, perpendiculars are drawn on the sides of an inscribed polygon; show that the polygons formed by joining the feet of the perpendiculars are (1) similar, (2) to one another as the distances of their generating points from the circle's centre.

Note by T. A. Finch, M.A.

This is incorrect in every case, except for a triangle, in which case it becomes Mr. McCay's well-known theorem. The Proposer seems to have overlooked the fact that *all* corresponding lines in similar figures must be in the same ratio; that this condition is not fulfilled, can be easily seen by taking the ratios of the lines joining the feet of the perpendiculars from the inverse points on any two sides of the polygon which do not intersect on the circle.

7389. (By C. Leudesdorf, M.A.)—If O, I are the centres, R, r the radii, of the circumscribed and inscribed circles of a spherical triangle ABC, and P any point on the sphere; prove that

$$\cos \text{IP} = \frac{\cos \text{OI}}{\cos \text{R}} - \frac{\cos r}{\sin \frac{1}{2}(a+b+c)} \left[\sin a \sin^2 \tfrac{1}{2}(\text{AP}) + \sin b \sin^2 \tfrac{1}{2}(\text{BP}) + \sin c \sin^2 \tfrac{1}{2}(\text{CP})\right].$$

Solution by D. Edwardes; Sarah Marks; *and others.*

If α, β, γ be the angles subtended by the sides at the pole of the circumscribed circle, and P any point on the sphere, it is easy to show that, since $\alpha+\beta+\gamma = 2\pi$,

$$\cos \text{PA} \sin \alpha + \cos \text{PB} \sin \beta + \cos \text{PC} \sin \gamma = 4 \cos \text{OP} \cos \text{R} \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma.$$

Applying this to the triangle DEF formed by joining the points of contact of the inscribed circle, and putting α, &c. for the angle FIE, &c., we have

$$\cos \text{PD} \sin \alpha + \cos \text{PE} \sin \beta + \cos \text{PF} \sin \gamma = 4 \cos \text{IP} \cos r \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma.$$

Now $\alpha = 2\text{AIE}$; whence, by the right-angled triangle AIE, we get

$$\sin \alpha = 2 \cos (s-a) \sin (s-a) \sin (s-b) \sin (s-c) \div \sin r \sin b \sin c,$$

and similarly for β and γ. Also

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma = \frac{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}{\sin r \cos^2 r \sin a \sin b \sin c}, \text{ since } \tan r = \frac{n}{\sin s};$$

hence $\cos \text{PD} \sin a \cos (s-a) + \cos \text{PE} \sin b \cos (s-b) + \cos \text{PF} \sin c \cos (s-c)$

$$= \frac{2 \cos \text{IP} \sin s}{\cos r}.$$

But, from the triangle PBC, we have

$$\cos \text{PD} \sin a = \cos \text{PB} \sin (s-c) + \cos \text{PC} \sin (s-b) \text{ \&c.}$$

Hence $\quad \cos \text{PA} \sin a + \cos \text{PB} \sin b + \cos \text{PC} \sin c = \dfrac{2 \cos \text{IP} \sin s}{\cos r}.$

Also (Todhunter's *Spherical Trigonometry*, Art. 144),

$$\sin a + \sin b + \sin c = \frac{2 \cos \text{OI} \sin s}{\cos \text{R} \cos r},$$

$$\therefore \frac{\cos \text{OI}}{\cos \text{R}} - \frac{\cos r}{\sin s}\left[\sin a \sin^2 \tfrac{1}{2}\text{AP} + \sin b \sin^2 \tfrac{1}{2}\text{BP} + \sin c \sin^2 \tfrac{1}{2}\text{CP}\right] = \cos \text{IP}.$$

7567. (By Professor SYLVESTER, F.R.S.)—Let nine quantities be supposed to be placed at the nine inflexions of a cubic curve, then they will group themselves in twelve sets of triads, which may be called collinear, and the product of each such triads may be called a collinear product. From the sum of the cubes of the nine quantities subtract three times the sum of their twelve collinear triadic products, and let the function so formed be called F. With another set of nine quantities form a similar function, say F′. Prove that FF′ will be also a similar function of nine quantities which will be lineo-linear functions of the other two sets, and find their values. [The inflexion-points are only introduced in order to make clear the scheme of the triadic combinations, so that the imaginariness of six of them will not matter to the truth of the theorem.]

Solution by R. RUSSELL, B.A.

The expression denoted by F may symmetrically be written down thus:—

$$F \equiv a^3+b^3+c^3+l^3+m^3+n^3+x^3+y^3+z^3-3abc-3lmn-3xyz-3alx-3bmy$$
$$-3cnz-3a(mz+ny)-3b(nx+lz)-3c(ly+mx) \quad \ldots\ldots\ldots\ldots(1).$$

A little consideration suggests the transformation (where $w^3 = 1$)

$$\left.\begin{array}{lll} a+b+c=\alpha, & l+m+n=\lambda, & x+y+z=\xi \\ a+bw+cw^2=\beta, & l+mw+nw^2=\mu, & x+yw+zw^2=\eta \\ a+bw^2+cw=\gamma, & l+mw^2+nw=\nu, & x+yw^2+zw=\zeta \end{array}\right\} \ldots(2),$$

which reduces F to the very simple form,

$$\alpha\beta\gamma+\lambda\mu\nu+\xi\eta\zeta-\alpha\lambda\xi-\beta\mu\eta-\gamma\nu\zeta \quad \text{or} \quad F = \begin{vmatrix} \alpha, & \nu, & \eta \\ \zeta, & \beta, & \lambda \\ \mu, & \xi, & \gamma \end{vmatrix},$$

and proves the remarkable property that any determinant of the third order can be reduced linearly, as in (2), to the form of expression given in the question.

F′ can be expressed similarly; and FF′, being also a determinant of the third order whose constituents are lineo-linear functions of the two original sets of nine letters, can by the above be reduced at once to (1).

[The theorem is *the* analogue to EULER's theorem that the product of one sum of 4 squares by another is also a sum of 4 squares. In a precisely similar way, any determinant of the second order is reducible to a sum of 4 squares, with the aid of $\omega^2 = -1$.]

4925. (By the late Professor CLIFFORD, F.R.S.)—Let U, V, W = 0 be the point equations, and $u, v, w = 0$ the plane-equations of three quadrics inscribed in the same developable, and let $u+v+w$ be identically zero. Then, if a tangent plane to U, a tangent plane to V, and a tangent plane to W, are mutually conjugate in respect of $au+bv+cw = 0$, they will intersect on $\dfrac{U}{(b-c)^2}+\dfrac{V}{(c-a)^2}+\dfrac{W}{(a-b)^2}=0$, which passes through the curves of contact of the developable with $au+bv+cw$ and one other quadric.

Solution by W. J. C. SHARP, M.A.

Let $\alpha_1 x+\beta_1 y+\gamma_1 z+\delta_1 w=0,\ \alpha_2 x+\beta_2 y+\gamma_2 z+\delta_2 w=0,$ and $\alpha_3 x+\beta_3 y+\gamma_3 z+\delta_3 w=0$, be the three tangent planes, and let (U), (V), (W) denote the values of U, V, and W, when the coordinates of the intersection of the planes are substituted for (x, y, z, w). Also let u_{11}, u_{22}, u_{33}, &c., denote the values of u, &c., when $(\alpha_1, \beta_1, \gamma_1, \delta_1)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2)$, $(\alpha_3, \beta_3, \gamma_3, \delta_3)$ are substituted for $\alpha, \beta, \gamma, \delta$; and $u_{12}, u_{23}, u_{31}, v_{12}$, &c., the conditions that these planes should be conjugate with respect to u, v, &c. Then, by the conditions, we have

$$\left.\begin{array}{c} u_{11}=0,\ v_{22}=0,\ w_{33}=0,\ au_{12}+bv_{12}+cw_{12}=0 \\ au_{23}+bv_{23}+cw_{23}=0,\quad au_{31}+bv_{31}+cw_{31}=0 \\ u_{11}+v_{11}+w_{11}=0,\ u_{22}+v_{22}+w_{22}=0,\ u_{33}+v_{33}+w_{33}=0 \\ u_{12}+v_{12}+w_{12}=0,\ u_{23}+v_{23}+w_{23}=0,\ u_{31}+v_{31}+w_{31}=0 \end{array}\right\}\ldots(\text{A}),$$

and $u_{11}u_{22}u_{33}+2u_{12}u_{23}u_{31}-u_{11}u_{23}^2-u_{22}u_{31}^2-u_{33}u_{12}^2=\Delta^2(\text{U})=(\text{U})$ if $\Delta=1$,

$$v_{11}v_{22}v_{33}+2v_{12}v_{23}v_{31}-v_{11}v_{23}^2-v_{22}v_{31}^2-v_{33}v_{12}^2=\Delta'^2(\text{V})=(\text{V})$$

if $\Delta'=1$; and if $\Delta''=1$, then

$$w_{11}w_{22}w_{33}+2w_{12}w_{23}w_{31}-w_{11}w_{23}^2-w_{22}w_{31}^2-w_{33}w_{12}^2=\Delta''^2(\text{W})=(\text{W}).$$

But, from the equations (A), we have

$$(c-a)^2u_{33}u_{12}^2+(b-c)^2v_{33}v_{12}^2=0,$$

$$(a-b)^2v_{11}v_{23}^2+(c-a)^2w_{11}w_{23}^2=0 \text{ and } (b-c)^2w_{22}w_{31}^2+(a-b)^2u_{22}u_{31}=0,$$

and $$2u_{12}u_{23}u_{31}-u_{22}u_{31}^2-u_{33}u_{12}^2=(\text{U}),$$

$$2v_{12}v_{23}v_{31}-v_{11}v_{23}^2-v_{33}v_{12}^2=(\text{V}),\quad 2w_{12}w_{23}w_{31}-w_{11}w_{23}^2-w_{22}w_{31}^2=(\text{W}),$$

$$\therefore\ \frac{(\text{U})}{(b-c)^2}+\frac{(\text{V})}{(c-a)^2}+\frac{(\text{W})}{(a-b)^2}=2\left\{\frac{u_{12}u_{23}u_{31}}{(b-c)^2}+\frac{v_{12}v_{23}v_{31}}{(c-a)^2}+\frac{w_{12}w_{23}w_{31}}{(a-b)^2}\right\}=0,$$

since $u_{12}:v_{12}:w_{12}=u_{23}:v_{23}:w_{23}=\&\text{c.}=b-c:c-a:a-b.$

And if T and T′ be the ordinary covariants of U and V (Δ and Δ' still being $=1$), the point equation to $u+\lambda v=0$ is

$$\text{U}+\lambda\text{T}+\lambda^2\text{T}'+\lambda^3\text{V}=0\ \ldots\ldots\ldots\ldots(\text{B}),$$

and, since $u+v+w=0$, $-\text{W}=\text{U}+\text{T}+\text{T}'+\text{V}$; also, at the points where $u+\lambda v=0$ touches the developable, the equation (B) gives equal values for λ, and therefore $\text{T}+2\lambda\text{T}'+3\lambda^2\text{V}=0$ and $3\text{U}+2\lambda\text{T}+\lambda^2\text{T}'=0$, and therefore along the curve of contact

$$\begin{vmatrix} \text{U}+\text{V}+\text{W}, & 3\lambda^2\text{V}, & 3\text{U} \\ 1, & 1, & 2\lambda \\ 1, & 2\lambda, & \lambda^2 \end{vmatrix}=0,\ \text{ or }\ -3\lambda^2(\text{U}+\text{V}+\text{W})+3\lambda^2\text{V}(2\lambda-\lambda^2)+3\text{U}(2\lambda-1)=0,$$

$$\text{or}\ \ \text{U}(\lambda-1)^2+\text{V}\lambda^2(\lambda-1)^2+\text{W}\lambda^2=0,$$

and this is identical with $\dfrac{\text{U}}{(b-c)^2}+\dfrac{\text{V}}{(c-a)^2}+\dfrac{\text{W}}{(a-b)^2}=0,$

if $\lambda^2:1:(\lambda-1)^2=(b-c)^2:(c-a)^2:(a-b)^2$, or $\lambda=\dfrac{b-c}{a-c}$,

and $u+\lambda v=0$ the same as $au+bv+cw=0$.

7571. (By Professor HAUGHTON, F.R.S.)—A solid body is bounded by two infinite parallel planes kept constantly at the temperature of melting ice, and by a third plane, perpendicular to the first two planes, kept constantly at the temperature of boiling water. After the lapse of a very long time, show that the law of distribution of temperatures will be represented by the equations (between the limits $y = \pm \frac{1}{2}\pi$)

$$v = ae^{-x}\cos y + be^{-3x}\cos 3y + \&c., \quad 1 = a\cos y + b\cos 3y + \&c.$$

Solution by T. WOODCOCK, B.A.; Prof. NASH, M.A.; *and others.*

We have to find v in terms of x and y, knowing $\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0$, also $v = 0$ when $y = \pm \frac{1}{2}\pi$, and $v = 1$ when $x = 0$, as well as when $x = \infty$.

Try $v = \Sigma\, u\cos ry$, where u is independent of y. We must have r = an odd integer $= 2m+1$ say, and also $\frac{d^2u}{dx^2} - r^2u = 0$; $\therefore u = Ae^{rx} + ae^{-rx}$. Now, when x is infinite, $v = 0$; $\therefore A = 0$; $\therefore v = \Sigma ae^{-(2m+1)x}\cos(2m+1)y$, the summation extending over all positive integral values of m. Putting $x = 0$, we have $1 = \Sigma a\cos(2m+1)y$, the limits of y being $\pm\frac{1}{2}\pi$.

7579. (By R. A. ROBERTS, M.A.)—Two uniform spherical shells attract according to the law of the inverse fifth power of the distance; show that, if they cut orthogonally, they will be in equilibrium under the influence of their mutual attraction.

Solution by Prof. TOWNSEND, F.R.S.; J. A. OWEN, B.Sc.; *and others.*

The attraction, for the law of the inverse fifth power of the distance, of a thin uniform spherical shell, upon a particle in its space, either external or internal to its mass, being directed towards the point of its surface nearest to the particle, and varying directly as the radial distance from its centre and inversely as the sixth power of the tangential distance from its surface; it follows at once that, for two such shells intersecting at right angles in a common space, if an elementary cone be supposed to diverge from the centre of either, it will intercept on the surface of the other two elements of mass, whose attractions by the former pass in opposite directions through its centre, and are to each other directly as the cubes of the radial distances from its centre, and inversely as the sixth powers of the tangential distances from its surface—that is, directly as the cubes of the radial distances from its centre, and inversely as the cubes of the perpendicular distances from its plane of intersection with the latter; and the two attractions being consequently equal in magnitude and opposite in direction, therefore, &c., as regards the property in question.

7569. (By Professor TOWNSEND, F.R.S.)—In a tetranodal cubic surface in a space, show that—

(*a*) The four nodal tangent cones envelope a common quadric.
(*b*) Their four conics of intersection with the opposite faces of the nodal tetrahedron lie in a common quadric.

(*c*) The two aforesaid quadrics envelope each other along a plane having triple contact with the surface.

Solution by Professor MALET, F.R.S.; Prof. NASH, M.A.; *and others.*

The vertices of the tetrahedron of reference being the four nodes, the equation of the cubic is of the form $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}+\frac{d}{w}=0$; and the equations of the four nodal tangent cones are

$$\frac{xy}{ab}+\frac{xz}{ac}+\frac{yz}{bc}=0,\qquad \frac{xy}{ab}+\frac{xw}{ad}+\frac{yw}{bd}=0,$$

$$\frac{xz}{ac}+\frac{xw}{ad}+\frac{zw}{cd}=0,\qquad \frac{yz}{bc}+\frac{yw}{bd}+\frac{zw}{cd}=0;$$

which envelope the quadric

$$\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d}\right)^2-4\left(\frac{xy}{ab}+\frac{xz}{ac}+\frac{yz}{bc}+\frac{xw}{ad}+\frac{yw}{bd}+\frac{zw}{cd}\right)=0,$$

and each of which meets the opposite face of the tetrahedron of reference in a conic situated on the quadric $\frac{xy}{ab}+\frac{xz}{ac}+\frac{yz}{bc}+\frac{xw}{ad}+\frac{yw}{bd}+\frac{zw}{dc}=0$. Now the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{w}{d}=0$ is a triple tangent plane to the cubic, the points of contact being $(a,\ b,\ -c,\ -d)$, $(a,\ -b,\ -c,\ d)$, $(a,\ -b,\ c,\ -d)$; therefore, &c.

NOTE ON INVERSE-COORDINATE CURVES, WITH SOLUTION OF QUEST. 6969. By R. TUCKER, M.A.

The general discussion of the properties of these curves is given in SALMON'S *Higher Curves* (1st ed., p. 238, &c.; 2nd ed., p. 244). We have $xx'=yy'=c^2$, whence $\tan\theta=\cot\theta'$, *i.e.* $\theta+\theta'=\frac{1}{2}\pi$ and $rr'\sin 2\theta=2c^2$ (1, 2).

If $y=mx$, then i.-c. c. [*i.e.* inverse-coordinate curve] is $x=my$, *i.e.* a line through the origin has for i.-c. c. a line through the origin equally inclined to other axis (rectangular axes). Therefore, if any number of points on one curve are collinear with the origin, there will be the same number on the i.-c. c. collinear with the origin.

If $y=mx+b$, then i.-c. c. is $bxy=c^2(x-my)$. Hence, if parallel chords be drawn to the primitive curve, the corresponding points on "i.-c." lie on hyperbolas, the locus of whose centres is a straight line through the origin orthogonal to the above system of chords.

If ψ has the usual meaning, then $\cot\psi'\sim\cot\psi=\cot 2\theta\sim\cot 2\theta'=2\cot 2\theta$.

This enables us at once to solve the following Question (6969):— "If, in a parabola vertical chords AP, AP', complementally inclined to the axis, make angles ψ, ψ' with the tangents at P, P',

then $$\cot\psi'\sim\cot\psi=2\Delta APP'/L^2."$$

The parabola $y^2 = 4ax$ will be its own "inverse" if $c = 4a = L$, then $2\Delta APP' = AP.AP' \sin PAP' = rr' \cos 2\theta = 2c^2 \cot 2\theta = 32a^2 \cot 2\theta$

$$= L^2 (2 \cot 2\theta) = L^2 (\cot\psi' \cot\psi).$$

Similar properties are readily obtained for other curves which are their own i.-c. c. as $2xy = a^2$ $(a^2 = 2c^2)$, $x^3 = ay^2$ $(c = a)$, &c.

[Another solution of Quest. 6969 is given on p. 114 of Vol. 37 of the *Reprint.*]

7194. (By Professor WOLSTENHOLME, M.A., Sc.D.)—In the examination for the Mathematical Tripos, January 2, 1868, Question (6) is as follows:—"If there be n straight lines lying in one plane so that no three meet in one point, the number of groups of n of their points of intersection, in each of which no three points lie in one of the n straight lines, is $\frac{1}{2}(n-1)$." Prove that this is not true; but that, if "n-sided polygons" be written for "groups of n points, &c.," the result will be true: and calculate the correct answer to the question enunciated.

Solution by W. J. GREENSTREET, B.A.; A. MACMURCHY, B.A.; *and others.*

Denote the n straight lines by $1.2.3...n$. Make a group of n intersections in this way:—1 and 2, 2 and 3, ..., $n-1$ and n, n and 1. Then there are two, and only two, points on each straight line. Hence we must take two points on each straight line, for if not there would be more than two on some line or lines. So that we merely require now the number of ways the n intersections may be arranged in a ring, that is $\frac{1}{2}(n-1)!$

7247. (By Dr. CURTIS.)—Two magnets, whose intensities are I_1, I_2, and lengths a_1, a_2, are rigidly connected so as to be capable of moving only in a horizontal plane round a vertical line, which passes through the middle point of the line connecting the two poles of each magnet; if 2α denote the angle between the lines of poles of the two magnets in the direction of opposite poles, while θ denotes the inclination to the magnetic meridian of the line bisecting this angle, prove that (1) the positions of *stable* and *unstable* equilibrium (discriminating between them) are given by $\tan\theta = (I_1a_1 + I_2a_2)\tan\alpha / (I_1a_1 - I_2a_2)$; and hence (2), if the intensities of the two magnets be inversely proportional to their lengths, the positions of equilibrium will be such that the lines of poles of the magnets will be equally inclined to the magnetic meridian.

Solution by W. M. COATES, B.A.; BELLE EASTON; *and others.*

The moment of the first magnet is I_1a_1, and the angle its axis makes with the magnetic meridian $= \theta - \alpha$; therefore the moment of the couple tending to turn it is proportional to $I_1a_1 \sin(\theta - \alpha)$.

Similarly the moment of the couple tending to turn the second magnet in the opposite direction is proportional to $I_2a_2 \sin(\theta + \alpha)$. Hence, when the system is in equilibrium, $I_1a_1 \sin(\theta - \alpha) = I_2a_2 \sin(\theta + \alpha)$, whence $\tan\theta = (I_1a_1 + I_2a_2 \tan\alpha) / (I_1a_1 - I_2a_2)$. [Whether the sign of the right-hand side of this equation be positive or negative, as the angle θ is sought from its tangent, there will be two solutions θ_1, θ_2, whose difference is 180°. The *stable* position is that in which the magnetic axis of each needle, in the direction of its *north-seeking* pole, makes an acute angle with the meridional line in its *northerly* direction, as in such position, if the system be turned through a small angle, the moment tending to return it to its original position is increased, and the opposite one diminished. The *unstable* position is derivable from the *stable* by turning the system through 180°.] If the condition $I_1a_1 = I_2a_2$ be fulfilled, $\theta = 90°$, from which it follows at once that the axes of the magnets are equally inclined to the meridian.

7508. (By Professor SYLVESTER, F.R.S.)—If m, n be any two square matrices of the same order $M = (mn-nm)^2$,

$$N = (m^2n-nm^2)(n^2m-mn^2)-(n^2m-mn^2)(m^2n-nm^2),$$

$$P = \begin{vmatrix} m^2, & mn+nm, & n^2 \\ m^2, & mn+nm, & n^2 \\ m^2, & mn+nm, & n^2 \end{vmatrix}; \text{ and D the determinant to the matrix } \alpha M+\beta N+\gamma P:$$

prove that D is an invariant to m, n; that is, remains unaltered when (supposing $pq'-p'q = 1$) $pm+qn$ and $p'm+q'n$ are substituted for m and n.

Solution by the PROPOSER.

Let m become $m+\epsilon n$ where ϵ is infinitesimal; then the increment of P divided by ϵ is

$$\begin{vmatrix} mn+nm, & mn+nm, & n^2 \\ mn+nm, & mn+nm, & n^2 \\ mn+nm, & mn+nm, & n^2 \end{vmatrix} + \begin{vmatrix} m^2, & 2n^2, & n^2 \\ m^2, & 2n^2, & n^2 \\ m^2, & 2n^2, & n^2 \end{vmatrix}, \text{ i.e., } 0.$$

Again, for N, call $(m^2n-nm^2) = A$, $(n^2m-mn^2) = B$; then $N = AB-BA$,

$$\text{and} \quad \frac{\delta A}{\epsilon} = (mn+nm)\,n - n\,(mn+nm) = B, \quad \frac{\delta B}{\epsilon} = n^2.n - n.n^2 = 0;$$

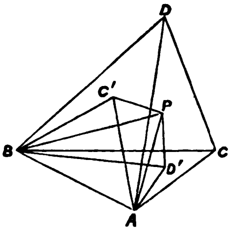
$$\text{hence} \qquad \delta N = \delta A.B - B\delta A = (B^2-B^2)\,\delta\epsilon = 0.$$

Finally, $\delta\,(mn-nm) = \epsilon\,(n-n) = 0$; hence $\alpha M+\beta N+\gamma P$ is unaltered by the change of m into $m+\epsilon n$ and in like manner of n into $n+\epsilon n$; whence it follows, by the same reasoning as in the theory of ordinary invariance, that m and n may be changed into $pm+qn$ and $p'm+q'n$, provided $pq'-p'q = 1$ without $\alpha M+\beta N+\gamma P$ undergoing a change, and consequently without its determinant changing, so that this latter is a binary invariant of the matrices m, n, as was to be pr0ved.

[Professor SYLVESTER calls attention to the immense new horizon in the theory of Invariants opened out by this question, which forms part of a general theory of Matrices, including the algebraical theory of Quaternions as an insignificant single case; and in which he connects the subject with the ordinary concomitants of Ternary as well as of Binary forms, and in such a manner as to *react* upon the ordinary theory of Invariants. The theorem in the question is part of the solution of the prodigiously difficult subject of Involution of Matrices, now happily accomplished, or brought at least within a stone's throw of accomplishment.]

7558. (By W. J. C. SHARP, M.A.)—If A′, B′, C′, D′, be the feet of the perpendiculars from any point on the four faces of a tetrahedron ABCD, show that $AC'^2 - BC'^2 = AD'^2 - BD'^2$, &c., and conversely.

Solution by W. G. LAX, B.A.; MARGARET T. MEYER; *and others.*



Join AP, BP, where P is the point from which the perpendiculars are drawn; then, since PC′B and PC′A are right angles,

$$AC'^2 - BC'^2 = AP^2 - PC'^2 - BP^2 + PC'^2$$
$$= AP^2 - BP^2.$$

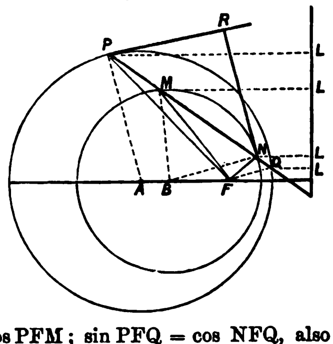
Similarly, $AD'^2 - BD'^2 = AP^2 - BP^2$,

therefore $AC'^2 - BC'^2 = AD'^2 - BD'^2$,

and so on, and conversely.

7364. (By W. S. M'CAY, M.A.)—If the line joining two points on two circles subtend a right angle at a limiting point, prove that the locus of the intersection of tangents at the points is a coaxal circle.

Solution by T. A. FINCH; SARAH MARKS; *and others.*



Let A and B be two circles, PN the chord subtending the right angle at F. Let PN meet the circles again in M and Q, and draw PL, ML, NL and QL perpendicular to L the radical axis of the circles and complete the Figure. Then

$$\frac{RP}{RN} = \frac{\sin RNP}{\sin RPN} = \frac{\cos BNM}{\cos APQ}$$
$$= \frac{MN}{2BM} \cdot \frac{2AP}{PQ}$$
$$= \frac{FM \,.\, FN \sin MFN}{FP \,.\, FQ \sin PFQ} \cdot \frac{AP}{BM},$$

and, from right angle, $\sin MFN = \cos PFM$; $\sin PFQ = \cos NFQ$, also $\angle PFM = \angle NFQ$ (since angles MFN and PFQ have same bisectors) therefore

$$\frac{RP^2}{RN^2} = \frac{FM^2 \,.\, FN^2}{FQ^2 \,.\, FP^2} \cdot \frac{AP^2}{BM^2} = \frac{BF \,.\, ML \,.\, BF \,.\, NL}{AF \,.\, QL \,.\, AF \,.\, PL} \cdot \frac{AP^2}{BM^2} = \frac{BF^2 \,.\, AF^2}{AP^2 \,.\, BM^2} = \text{const.}$$

Therefore, &c. [The PROPOSER remarks that this proof is much more elegant than his own. The theorem was originally derived by reciprocation from Question 5395, solved in *Reprint*, Vol. xxix., p. 23.]

1945. (By the late C. W. MERRIFIELD, F.R.S.)—Find a rectangular parallelepiped such that its edges, the diagonals of its faces, and the diagonals of the solid, shall all be integral.

Solution by ASÛTOSH MUKHOPÂDHYÂY.

Let x, y, z be the edges of the solid; then the diagonals of its faces are $(x^2+y^2)^{\frac{1}{2}}$, $(y^2+z^2)^{\frac{1}{2}}$, $(z^2+x^2)^{\frac{1}{2}}$; also, the diagonals of the solid are equal to one another, and represented by $(x^2+y^2+z^2)^{\frac{1}{2}}$. We have, accordingly, to investigate whether it is possible to find positive integral values of x, y, z, which make $x, y, z, (x^2+y^2)^{\frac{1}{2}}, (y^2+z^2)^{\frac{1}{2}}, (z^2+x^2)^{\frac{1}{2}}, (x^2+y^2+z^2)^{\frac{1}{2}}$ all integral. Let $x^2+y^2 = (k^2+1)^2$, $y^2+z^2 = (l^2+1)^2$..................(1, 2),

$$z^2+x^2 = (m^2+1)^2,\quad x^2+y^2+z^2 = (n^2+1)^2 \text{ ....(3, 4).}$$

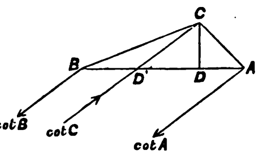
Then it is well-known that the solutions of (1, 2, 3) are

$$\left.\begin{matrix} x = 2k \\ y = k^2-1 \end{matrix}\right\} \left.\begin{matrix} y = 2l \\ z = l^2-1 \end{matrix}\right\} \left.\begin{matrix} z = 2m \\ x = m^2-1 \end{matrix}\right\}.$$

Now, substituting in (4) from (2), we get $x^2+(l^2+1)^2 = (n^2+1)^2$. Therefore $x = 2n$, $l^2+1 = n^2-1$, therefore $n^2-l^2 = 2$; hence the solution of the original problem depends on an equation of the form $(x+y)(x-y) = 2$. Now, a moment's consideration shows that this equation has no positive integral solution; for, assuming x, y to be positive integers, and since $(x+y)(x-y) = 2$, $(x+y)$, $(x-y)$ must be each a positive integral; and since the composition of 2 is unique (2×1), we must have $x+y = 2$, $x-y = 1$, which give $x = \frac{3}{2}$, $y = \frac{1}{2}$,—fractional values. Hence, it is demonstrated that the original system of equations has no positive integral solutions, and it is impossible to find the rectangular parallelepiped in question.

7573. (By Professor HUDSON, M.A.)—Parallel forces act at the angular points of a triangle proportional to the cotangents of the angles. Can they be in equilibrium?

Solution by (1) B. REYNOLDS, M.A.; (2) W. J. BARTON, B.A.



1. Yes: if $a^2+b^2 = 3c^2$, and if

$$BD' = DA = b\cos A,$$

CD′ being the direction of the forces.

For $\cot C = \cot A + \cot B$,

$$\therefore\ \frac{\cot C}{\sin C} = \frac{\sin C}{\sin A \sin B}, \text{ or } \cos C = \frac{c^2}{ab},$$

or $2ab\cos C = 2c^2$, whence $a^2+b^2 = 3c^2$.

Next, if $BD' = x$, we have $x\cot B = (c-x)\cot A$,

$$\text{or } x = c\,\frac{\cot A}{\cot A + \cot B} = c\,\frac{\cos A \sin B}{\sin (A+C)} = c\,\frac{\cos A \sin B}{\sin C} = b\cos A.$$

2. For equilibrium, one of the forces (at A say) must act in the opposite direction to the other two, whence $\cot A = \cot B + \cot C$; also, if the direction of force at A produced backwards cut BC in D, then, taking moments, $DC : BD = \cot B : \cot C$; and, if these conditions are

fulfilled, there is equilibrium and not otherwise. A particular case is when one of the angles (B say) is a right angle, so that the force of B vanishes; then we must have A = C = $\frac{1}{4}\pi$, and forces at A and C act in opposite directions along the hypotenuse.

[Mr. REYNOLDS states that he "inadvertently made the angle BCA obtuse, whereas it should be acute, since $c^2 < a^2 + b^2$."

Professor HUDSON remarks that "the figure is manifestly impossible; for, C being obtuse, cot C is −, and a force cot C in the opposite direction to cot A, cot B, is really in the same direction and cannot counterbalance them.

"In the case supposed by Mr. BARTON, the forces are proportional to cot A, 0, −cot C. Since for equilibrium one of the forces must act in the opposite direction to the other two, the proper inference is that the triangle is obtuse-angled.

"The condition of equilibrium is $\cot A + \cot B + \cot C = 0$, therefore $\frac{\sin(A+B)}{\sin A \sin B} + \frac{\cos C}{\sin C} = 0$, therefore $\frac{\sin^2 C}{\sin A \sin B} + \frac{\sin^2 A + \sin^2 B - \sin^2 C}{2 \sin AFB} = 0$, **therefore $\sin^2 A + \sin^2 B + \sin^2 C = 0$, which is impossible].**"

7495. **(By S. TEBAY, B.A.)—Show that the mean length of the "Sailor's Knot," or geographical mile, in latitude λ, is approximately $1{\cdot}1566\,(1 - {\cdot}00667 \cos^2\lambda)$ mile.**

Solution by the PROPOSER.

If the ellipsoid $(1-e^2)(x^2+y^2) + z^2 = b^2$ be cut by a diametral plane passing through a point in latitude λ, and inclined to the plane of the meridian at an angle θ, the equation to the section is

$$(1 - e^2\cos^2\lambda)\,x^2 - 2e^2 \sin\lambda \cos\lambda \cos\theta\, xy + (1 - e^2 + e^2 \cos^2\lambda \cos^2\theta)\,y^2 = b^2.$$

Let this be written $\quad Ax^2 - 2Cxy + By^2 = b^2.$

Differentiate twice with respect to x; thus

$$Ax - Cy - Cxp + Byp = 0, \quad A - 2Cp - Cyq + Bp^2 + Byq = 0.$$

Hence at a point on the meridian, putting $y = 0$ and $x = \frac{b}{A^{\frac{1}{2}}}$, we have

$$p = \frac{A}{C}, \quad q = \frac{A^{\frac{3}{2}}}{bC^3}(AB - C^2); \quad \therefore\ \rho = \frac{(1+p^2)^{\frac{3}{2}}}{q} = \frac{\left(A + \frac{C^2}{A}\right)^{\frac{3}{2}}}{AB - C^2}\, b$$

$$= \frac{a^2}{b}\left\{\frac{1 - e^2(2-e^2)\cos^2\lambda}{1 - e^2\cos^2\lambda}\right\} \frac{\left\{1 - \frac{e^4 \sin^2\lambda \cos^2\lambda \sin^2\theta}{1 - e^2(\ \ - e^2)\cos^2\lambda}\right\}^{\frac{3}{2}}}{1 - e^2\cos^2\lambda \sin^2\theta}$$

$$= H\frac{a^2}{b}\,\frac{(1 - n\sin^2\theta)^{\frac{3}{2}}}{1 - m\sin^2\theta}, \text{ suppose,}$$

$$= H\frac{a^2}{b}\left\{1 + (m - \tfrac{3}{2}n)\sin^2\theta + \left(m^2 - \tfrac{3}{2}mn + \frac{1}{2!}\tfrac{3}{2}n^2\right)\sin^4\theta\right.$$

$$\left. + \left(m^3 - \tfrac{3}{2}m^2n + \frac{1}{2!}\tfrac{3}{2}\tfrac{1}{2}mn^2 + \frac{1}{3!}\tfrac{3}{2}\tfrac{1}{2}\tfrac{1}{2}n^3\right)\sin^6\theta + \&c.\right\}.$$

Let this be written $\rho = \text{H}\,\frac{a^2}{b}\,(1 + \text{N}_2 \sin^2\theta + \text{N}_4 \sin^4\theta + \text{N}_6 \sin^6\theta + \dots)$;

then, since $\int_0^{\frac{1}{2}\pi} \sin^n\theta\, d\theta = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \dots \tfrac{1}{2}\,\tfrac{1}{2}\pi$, ($n$ even),

the mean value of ρ is

$$\rho' = \left(\int^{\frac{1}{2}\pi} \rho\, d\theta\right) \div \tfrac{1}{2}\pi = 4\,\frac{a^2}{b}\left(1 + \tfrac{1}{2}\text{N}_2 + \frac{3\,.\,1}{4\,.\,2}\,\text{N}_4 + \frac{5\,.\,3\,.\,1}{6\,.\,4\,.\,2}\,\text{N}_6 + \&\text{c.}\right)$$

$$= \frac{a^2}{b}\,(1 - e^2\cos^2\lambda + \tfrac{3}{4}e^4 \sin^2\lambda\cos^2\lambda + \tfrac{9}{16}e^6 \sin^2\lambda\cos^4\lambda + \dots), \text{ nearly.}$$

Thus the mean value of ρ at a point on the equator is b. Take

$$a = 3962{\cdot}824, \quad b = 3949{\cdot}585 \text{ miles};$$

then the mean length of the knot at a point on the equator is

$$\frac{\pi b}{10800} = 1{\cdot}14889 \text{ mile} = 1 \text{ m. } 1 \text{ fur. } 42 \text{ yds.}$$

Approximately, the length of the knot is

$$\frac{\pi\rho'}{10800} = 1{\cdot}1566\,(1 - e^2\cos^2\lambda + \tfrac{3}{4}e^4\sin^2\lambda\cos^2\lambda + \tfrac{9}{16}e^6\sin^2\lambda\cos^4\lambda)$$

$$= 1{\cdot}1566\,(1 - {\cdot}00667\cos^2\lambda).$$

At 30°, 45°, 60° we have, respectively, $\rho' = 3956{\cdot}450$, $3962{\cdot}881$, $3969{\cdot}5$, and length of knot 1·1508, 1·1527, 1·1547 mile.

7581. (By C. Leudesdorf, M.A.)—If $\text{A} + \text{B} + \text{C} = 180°$,

$$(y - z)\cot\tfrac{1}{2}\text{A} + (z - x)\cot\tfrac{1}{2}\text{B} + (x - y)\cot\tfrac{1}{2}\text{C} = 0,$$

$$(y^2 - z^2)\cot\text{A} + (z^2 - x^2)\cot\text{B} + (x^2 - y^2)\cot\text{C} = 0;$$

prove that $\dfrac{y^2 + z^2 - 2yz\cos\text{A}}{\sin^2\text{A}} = \dfrac{z^2 + x^2 - 2zx\cos\text{B}}{\sin^2\text{B}} = \dfrac{x^2 + y^2 - 2xy\cos\text{C}}{\sin^2\text{C}}$.

Solution by W. J. Barton, B.A.; Margaret T. Meyer; *and others.*

Let a, b, c be the sides of a triangle having opposite angles equal to A, B, C; then we have

$$\Sigma\,(b - c)\cot\tfrac{1}{2}\text{A} \propto \Sigma\,(\sin\text{B} - \sin\text{C})\cot\tfrac{1}{2}\text{A} \propto \Sigma\sin\tfrac{1}{2}(\text{B} - \text{C})\cos\tfrac{1}{2}\text{A}$$

$$\propto \Sigma\sin\tfrac{1}{2}(\text{B} - \text{C})\sin\tfrac{1}{2}(\text{B} + \text{C}) \propto \Sigma\,[\sin\tfrac{1}{2}(2\text{B}) - \sin\tfrac{1}{2}(2\text{C})] = 0 \quad \dots(1),$$

$$\Sigma\,(b^2 - c^2)\cot\text{A} \propto \Sigma\sin(\text{B} + \text{C})\sin(\text{B} - \text{C})\cot\text{A} \propto \Sigma\sin(\text{B} - \text{C})\cos(\text{B} + \text{C})$$

$$\propto \Sigma\,(\sin 2\text{B} - \sin 2\text{C}) = 0 \quad \dots(2);$$

therefore, from (1) and (2) combined with the given equations,

$$\frac{b - c}{y - z} = \frac{c - a}{z - x} = \frac{a - b}{x - y}, \text{ and } \frac{b^2 - c^2}{y^2 - z^2} = \frac{c^2 - a^2}{z^2 - x^2} = \frac{a^2 - b^2}{x^2 - y^2} \quad \dots(3),$$

therefore $\dfrac{a}{x} = \dfrac{b}{y} = \dfrac{c}{z}$; therefore a triangle can be constructed with x, y, z for sides and A, B, C for angles; hence we obtain

$$\frac{x^2}{\sin^2\text{A}} = \text{const.} = \frac{y^2 + z^2 - 2yz\cos\text{A}}{\sin^2\text{A}} = \frac{z^2 + x^2 - 2zx\cos\text{B}}{\sin^2\text{B}} = \&\text{c.}$$

3835. (By the EDITOR.)—The sides of a triangle ABC are BC = 6, CA = 5, AB = 4; and Q, R are points in AC, AB, such that CQ = 2; BR = 3. Show (1) by a general solution, that the distance from B to a point P in BC, such that $\angle$ CQP = BRP, is BP = $\frac{1}{3}(601^{\frac{1}{2}} - 13) = 3{\cdot}83843$; and (2) give a construction for finding the point P.

Solution by D. BIDDLE.; Prof. MATZ, M.A., Ph.D.; *and others.*

1. Draw (Fig. 1) PM, QS parallel to AB, and PN, RT parallel to AC; then the triangles QMP, RNP will be similar, and PN : PM = NR : QM;

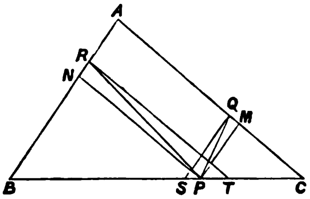


Fig. 1.

but $PN = \frac{AC \,.\, BP}{BC}$, $PM = \frac{AB \,.\, PC}{BC}$,

$NR = \frac{AB \,.\, PT}{BC}$, $QM = \frac{AC \,.\, SP}{BC}$;

therefore $\quad AC \,.\, BP : AB \,.\, PC = AB \,.\, PT : AC \,.\, SP,$

$$BP : PC = AB^2 \,.\, PT : AC^2 \,.\, SP;$$

but $PC = BC - BP$, $PT = \frac{BC \,.\, BR}{AB} - BP$, $SP = BP + \frac{CQ \,.\, BC}{AC} - BC$;

$\therefore BP : BC - BP = AB.BC.BR - AB\, BP : AC^2BP + AC.BC.CQ - AC^2BC,$

and $\quad BP^2 + \frac{AC \,.\, BC \,.\, CQ - AC^2BC + AB^2BC + AB \,.\, BC \,.\, BR}{AC^2 - AB^2} BP$

$$= \frac{AB \,.\, BC^2 \,.\, BR}{AC^2 - AB^2};$$

whence

$$2(AC^2 - AB\ )BP = \left\{ \begin{array}{l} (AC \,.\, BC \,.\, CQ - AC^2BC + AB^2BC + AB \,.\, BC \,.\, BR)^2 \\ \quad + 4(AC^2 - AB^2)\, AB \,.\, BC^2 \,.\, BR \end{array} \right\}^{\frac{1}{2}}$$
$$- (AC.BC \,.\, CQ - AC^2\, BC + AB^2\, BC + AB \,.\, BC \,.\, BR).$$

In the given case, $BP = \frac{1}{3}[(601^{\frac{1}{2}} - 13)] = 3{\cdot}83843.$

2. Draw (Fig. 2) QD perpendicular to BC; make DE = QD; join RE; make CF = BF (that is, from the mid-point of BC draw a perpendicular thereto). Draw EH parallel to BF, to meet AB in T; then TRE will be a triangle with an angle T = ABF = ABC − ACB, and the circle RPE, drawn round this triangle, will cut BC in the point P required, so that

$\angle$ CQP = BRP.

For $\angle$ SQD = SED = SRI,

$\angle$ CQD − BRI = ABC − ACB,

$\therefore$ SQC − BRS = ABC − ACB;

and, by construction, we have

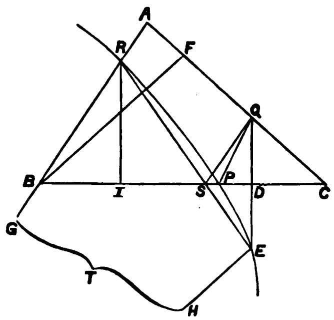


Fig. 2.

$\angle$ PER + PRE = ABF = ABC − ACB; but $\angle$ PER = PQS,

$\therefore$ $\angle$ PRE + PQS = ABC − ACB = SQC − BRS, and $\quad$ PRE + BRS = SQC − PQS,

hence $\quad$ ∠BRP = CQP.

[If BC = a, CA = b, AB = c, BR = m, CQ = n, and BP = x, we have

$$\cot \mathrm{BRP} = \frac{m - x\cos B}{x \sin B} = \cot \mathrm{CQP} = \frac{n-(a-x)\cos C}{(a-x)\sin C};$$

therefore $\frac{m}{x}$ cosec B $-\frac{n}{a-x}$ cosec C = cot B − cot C, which gives

the quadratic equation $\quad \frac{mc}{bx} - \frac{n}{a-x} = \frac{c\cos B - b\cos C}{b} = \frac{c^2-b^2}{ab}$;

whence, putting, for shortness' sake, $k = nb + mc - b^2 + c^2$, we obtain the

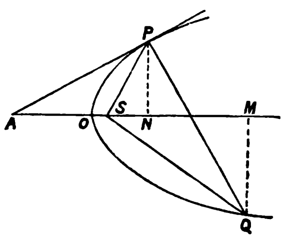
general value $\quad x = \frac{a}{2(c^2-b^2)}\left\{k \pm [k^2 - 4mc(c^2-b^2)]^{\frac{1}{2}}\right\}$,

which, with the numbers in the question, gives the specified result.]

7159. (By R. Knowles, B.A., L.C.P.)—In a parabola whose latus rectum is $4a$, if θ be the angle subtended at the focus S by a normal chord PQ, prove that the area of the triangle SPQ = $a^2 \cot \frac{1}{2}\theta (\tan \frac{1}{2}\theta + 4 \cot \frac{1}{2}\theta)^2$.

Solution by J. S. Jenkins; Sarah Marks; *and others.*

Let (x', y'), (x'', y'') be the coordinates of P and Q, and let θ = ∠PSQ; then it can be readily shown that



$$\mathrm{OM} = x'' = \frac{(x'+2a)^2}{x'},$$

$$\mathrm{QM} = y'' = 2\left(\frac{a}{x'}\right)^{\frac{1}{2}}(x'+2a),$$

$$\Delta\,\mathrm{PSQ} = 2\left(\frac{a}{x'}\right)^{\frac{1}{2}}(a+x')^2 \ldots (1);$$

$$\Delta\,\mathrm{PSQ} = \tfrac{1}{2}(a+x')(a+x'')\sin\theta = \tfrac{1}{2}(a+x')\left(\frac{x'^2+4a^2+5ax'}{x'}\right)\sin\theta \ldots (2);$$

$$\therefore\ \sin\theta = \frac{4(ax')^{\frac{1}{2}}(a+x')}{x'^2+4a^2+5ax'} = \frac{8ay'}{y'^2+16a^2};\ \therefore\ y' = 4a\cot\tfrac{1}{2}\theta, \text{ and } x' = 4a\cot^2\tfrac{1}{2}\theta.$$

This value of x' substituted in (1) gives the area

$$\Delta\,\mathrm{PQS} = 2\left(\frac{a}{4a\cot^2\frac{1}{2}\theta}\right)^{\frac{1}{2}}(a+4a\cot^2\tfrac{1}{2}\theta)^2 = a^2\cot\tfrac{1}{2}\theta\left(\frac{1+4\cot^2\frac{1}{2}\theta}{\cot\frac{1}{2}\theta}\right)^2$$

$$= a^2\cot\tfrac{1}{2}\theta\,(\tan\tfrac{1}{2}\theta + 4\cot\tfrac{1}{2}\theta)^2.$$

7578. (By the Rev. T. C. Simmons, M.A.)—If a number have the sum of its digits equal to 10, find under what circumstances twice the number will have the sum of its digits equal to 11.

Solution by MARGARET T. MEYER; C. W. H. GREAVES; *and others.*

If all the digits are under 5, the double number will clearly have the sum of its digits = 20. If two of the digits are 5 (the rest of course all = 0), the double number will have sum of digits = 2. In all other cases, the sum will be 11 (or 20 − 9). [This is included in Quest. 7534, solved on p. 28 of this volume.]

7542. (By Professor MARTIN, M.A., Ph.D.)—Prove that for $n = \infty$,

$$\frac{\pi}{2n}\left\{\frac{1}{1+\sqrt{2}\sin\left(\frac{1}{4}\pi+\frac{\pi}{2n}\right)}+\ldots+\frac{1}{1+\sqrt{2}\sin\left(\frac{1}{4}\pi+\frac{n\pi}{2n}\right)}\right\} = \log_e 2.$$

Solution by Professor WOLSTENHOLME, M.A., Sc.D.

$$\int_0^{\frac{1}{2}\pi}\frac{dx}{1+\sin x+\cos x} = \int_0^{\frac{1}{2}\pi}\frac{dx}{2\cos\frac{1}{2}x\,(\cos\frac{1}{2}x+\sin\frac{1}{2}x)}$$

$$= \int_0^{\frac{1}{2}\pi}\frac{\frac{1}{2}\sec^2\frac{1}{2}x\,dx}{1+\tan\frac{1}{2}x} = \left[\log(1+\tan\tfrac{1}{2}x)\right]_0^{\frac{1}{2}\pi} = \log 2.$$

Writing this in the usual way, $dx = \pi/2n$, and the limit of the series in the question, when n is ∞, is log 2.

Since $$\int_0^{\frac{1}{2}\pi}\frac{x}{1+\sin x+\cos x}\,dx = \int_0^{\frac{1}{2}\pi}\frac{\frac{1}{2}\pi - x}{1+\cos x+\sin x}\,dx,$$

we get $$\int_0^{\frac{1}{2}\pi}\frac{x}{1+\sqrt{2}\sin(\frac{1}{4}\pi+x)}\,dx = \tfrac{1}{4}\pi\log 2;$$

and this integral may be written

the limit of $$\frac{1}{n^2}\left\{\frac{1}{1+\sqrt{2}\sin\left(\frac{1}{4}\pi+\frac{\pi}{2n}\right)}+\frac{2}{1+\sqrt{2}\sin\left(\frac{1}{4}\pi+\frac{2\pi}{2n}\right)}+\ldots\right.$$

$$\left.\ldots+\frac{n}{1+\sqrt{2}\sin\left(\frac{1}{4}\pi+\frac{n\pi}{2n}\right)}\right\},$$

when n tends to ∞ is $\frac{1}{\pi}\log 2$. This is not new, being only another way of getting at the well-known definite integral

$$\int_0^{\frac{1}{4}\pi}\log(1+\tan x)\,dx = \tfrac{1}{8}\pi\log 2.$$

7592. (By S. TEBAY, B.A.)—Find an integral value of a such that $(m^2+n^2)^2+a$ and $(m^2+n^2)^2-a$ shall be rational squares; m and n being positive integers.

Solution by MORGAN JENKINS, M.A.

If $(m^2+n^2)^2+a = (h+k)^2$ and $(m^2+n^2)^2-a = (h-k)^2$,

then $a = 2h\kappa$, and $(m^2+n^2)^2 = h^2+\kappa^2$. One solution of the last equation is $h = m^2-n^2$, $\kappa = 2mn$, whence $a = 4mn(m^2-n^2)$; and

$$(m^2+n^2)^2 \pm 4mn(m^2-n^2) = (m^2-n^2\pm 2mn)^2.$$

The total number of primitive ways of expressing, as the sum of two different squares (zeros excluded), a number that contains no odd prime factors, and no power of 2 higher than 2^1, is 2^{t-1}, where t is the number of different odd prime factors which $\equiv 1$, mod. 4, and which are contained in the number. Hence the total number of solutions may be found when the mode of separating m^2+n^2 into factors is given; thus, if

$$m^2+n^2 = 2^\lambda . X^2 . \alpha^u . \beta^v . \gamma^w \ldots,$$

where X is an odd number all of whose factors are of the form $4p+3$, and $\alpha, \beta, \gamma \ldots$ are t prime factors, each of the form $4p+1$ (it being known that all the prime factors of X must be of even degree), then

$$(m^2+n^2)^2 = 2^{2\lambda} . X^4 . \alpha^{2u} . \beta^{2v} . \gamma^{2w} \ldots,$$

and, if $\mu^2+\nu^2$ be one of the 2^{t-1} primitive representations of $\alpha^{2u} . \beta^{2v} . \gamma^{2w} \ldots$ as the sum of two squares, $h = 2^\lambda . X^2\mu$ and $k = 2^\lambda . X^2\nu$. So, if $\mu^2+\nu^2$ be one of the 2^{t-1} primitive representations of $\alpha^{2u-2} . \beta^{2v} . \gamma^{2w} \ldots$ as the sum of two squares, we have $h = 2^\lambda . X^2\alpha\mu$, $k = 2^\lambda . X^2\alpha\nu$, and so on; but the number of primitive representations of $\alpha^0\beta^v\gamma^w \ldots$ will be only 2^{t-2}, and of $\alpha^0\beta^0\gamma^{2w} \ldots$ will be only 2^{t-3}. Thus the total number of solutions will be the sum of the coefficients in the product

$$(\tfrac{1}{2}+\alpha^2+\alpha^4+\ldots\alpha^{2u})(\tfrac{1}{2}+\beta^2+\beta^4+\ldots\beta^{2v})(\tfrac{1}{2}+\gamma^2+\gamma^4\ldots+\gamma^{2w})\ldots\times 2^{t-1},$$

omitting the first term in the product, viz., $\dfrac{2^{t-1}}{2^t}$ or $\tfrac{1}{2}$, if we reject the solution $a = 0$; and therefore the total number of solutions will be

$$2^{t-1}(u+\tfrac{1}{2})(v+\tfrac{1}{2})(w+\tfrac{1}{2})\ldots-\tfrac{1}{2} \text{ or } \tfrac{1}{2}\left[(2u+1)(2v+1)(2w+1)\ldots-1\right].$$

[Since $101 = 10^2+1^2$, if we take $m = 10$, $n = 1$, we have $a = 3960$, and $101^2+3960 = 119^2$, $101^2-3960 = 79^2$.]

7351. (By Professor SYLVESTER, F.R.S.)—Let ν be the number of ways in which any number n can be composed with any i positive integers (all unequal), and let X_i represent the sum of the terms νx^n, which will be an *infinite* series. Also, let ν_j be the number of ways in which any number n can be composed with any i positive integers all unequal as before, but now *none greater* than j, and let $X_{i,j}$ represent the sum of the terms x^n which will be a *finite* series. Prove that

$$X_{i,j} = (1-x^j)(1-x^{j-1})\ldots(1-x^{j-i+1})X_i.$$

Ex.—Let $i = 2$, $j = 3$; then

$$X_i = x^3+x^4+2x^5+2x^6+3x^7+3x^8+4x^9+\ldots\ldots$$

and $$X_{i,j} = x^3+x^4+x^5 = (1-x^2)(1-x^3)X_i.$$

Solution by W. J. C. SHARP, M.A.

1. Evidently X_i is the coefficient of z^i in the expansion of the infinite product $(1+xz)(1+x^2z)(1+x^3z)\ldots$, so that this is $1+X_1z+X_2z^2+X_3z^3+\&c.$; but, if $xz=y$, the same product

$= (1+y)(1+xy)(1+x^2y)(1+x^3y)\ldots = (1+y)(1+X_1y+X_2y^2+\&c.)$

$= (1+xz)(1+X_1xz+X_2x^2z^2+X_3x^3z^3+\&c.)$;

hence, equating coefficients, $X_1=(1+X_1)x$, $X_2=(X_1+X_2)x^2$, &c., &c.,

$X_i=(X_{i-1}+X_i)x^i$, so that $X_1=\dfrac{x}{1-x}$, &c.,

and
$$X_i=\frac{x^i}{1-x^i}X_{i-1}=\frac{x^{\frac{1}{2}i(i+1)}}{(1-x)(1-x^2)\ldots(1-x^i)}$$

[a value obtained in a different way in Art. 2]. Similarly $X_{i,j}$ is the coefficient of z^i in the expansion of $(1+xz)(1+x^2z)\ldots(1+x^jz)$, so that this product $=1+X_{1,j}z+X_{2,j}z^2+X_{3,j}z^3+\&c.$; and if, as before, $xz=y$, this

product $=\dfrac{1+y}{1+x^jy}(1+xy)(1+x^2y)\ldots(1+x^jy)$

$$=\frac{1+xz}{1+x^{j+1}z}\left\{1+X_{1,j}xz+X_{2,j}x^2z^2+X_{3,j}x^3z^3+\&c.\right\};$$

and, multiplying by $1+x^{j+1}z$ and equating coefficients,

$$x^{j+1}+X_{1,j}=(1+X_{1,j})x,\quad x^{j+1}X_{1,j}+X_{2,j}=(X_{1,j}+X_{2,j})x^2,\ \&c.,$$

$$x^{j+1}X_{i-1,j}+X_{i,j}=(X_{i-1,j}+X_{i,j})x^i,\ \text{so that } X_{1,j}=\frac{x(1-x^j)}{1-x}\ \&c.,$$

$$X_{i,j}=\frac{x^i(1-x^{j-i+1})}{1-x^i}X_{i-1,j}=x^{\frac{i(i+1)}{2}}\frac{(1-x^j)(1-x^{j-1})\ldots(1-x^{j-i+1})}{(1-x)(1-x^2)\ldots(1-x^i)}$$

$$=(1-x^j)(1-x^{j-1})\ldots(1-x^{j-i+1})X_i.$$

2. Evidently $\quad X_1=x+x^2+x^3+x^4+\&c.=\dfrac{x}{1-x}$,

and $\quad X_2=x^3+x^4+2x^5+2x^6+3x^7+3x^8+4x^9+\&c.$

$$=x^3(1+x)(1+2x^2+3x^4+4x^6+\&c.)=\frac{x^3(1+x)}{(1-x^2)^2}=\frac{x^3}{(1-x)(1-x^2)}.$$

Now $X_3=(x^3+x^6+x^9+\&c.)X_2$; for, if ν_n be the coefficient of x^n in X_3 and μ_n that in X_2, ν_n is made up of μ_{n-3}, the number of solutions of $p+q+r=n$ in which one value is 1 (p, q, and r being all unequal integers); of μ_{n-6}, the number of solutions in which one value is 2 and none 1, and so on. Therefore $X_3=\dfrac{x^3}{1-x^3}X_2=\dfrac{x^6}{(1-x)(1-x^2)(1-x^3)}$, and a repetition of the same argument leads to the result above stated.

7377. (By Professor SYLVESTER, F.R.S.)—Integrate the equation in differences $\quad u_{n+1}=u_n+n(n-1)u_{n-1}+(2n-1)\omega_n$, where ω_n denotes the product of n terms of the fluctuating progression

$$1, 1, 3, 3, 5, 5, 7 \ldots\ldots$$

Solution by W. J. C. SHARP, M.A.

The equation $u_{n+1} = u_n + n(n-1)\,u_{n-1}$ is remarkable, as, though it is of the second order, when solved by successive substitution it only involves one constant. The solution is $u_n = \omega_n a$. This is easily verified as follows:—

Let $n = 2p-1$, then $\omega_n a + n(n-1)\,\omega_{n-1} a = 1 \,.\, 1 \,.\, 3 \ldots 2p-3 \,.\, 2p-1 \,.\, a$

$$+ 2(p-1)(2p-1) \,.\, 1 \,.\, 1 \ldots 2p-3 \,.\, 2p-3 \,.\, a$$

$$= 1 \,.\, 1 \,.\, 3 \,.\, 3 \ldots 2p-3 \,.\, 2p-1 \,.\, 2p-1\, a = \omega_{n+1} a.$$

Let $n = 2p$, then $\omega_n a + n(n-1)\,\omega_{n-1} a = 1 \,.\, 1 \,.\, 3 \ldots 2p-1 \,.\, 2p-1 \,.\, a$

$$+ 2p(2p-1) \,.\, 1 \,.\, 1 \,.\, 3 \ldots 2p-1 \,.\, a$$

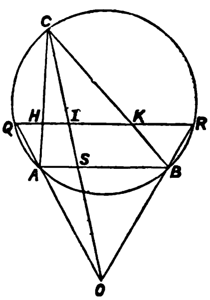
$$= 1 \,.\, 1 \,.\, 3 \,.\, 3 \ldots 2p-1 \,.\, 2p-1 \,.\, 2p+1 \,.\, a = \omega_{n+1} a.$$

The solution of the given equation may therefore be put in the form $u_n = \omega_n a + v_n$, where v_n is the value of u_n obtained by putting $a = u_1 = 0$. Then $v_1 = 0$, $v_2 = 1$, $v_3 = 4$, $v_4 = 25$, $v_5 = 136$, $v_6 = 1041$, $v_7 = 7596$, &c. I have been unable to find a functional expression for v_n.

[The reduction in the number of constants only applies when n, &c. are integers, and seems to be due to the fact that, for such values, two successive equations, $(n = 1)\; u_2 = u_1$, $(n = 0)\; u_1 = u_2$, are of the first order.]

7544. (By the EDITOR.)—Construct a triangle, having given the base, the vertical angle, and the ratio of the segments of a given chord of the circumscribed circle drawn parallel to the base, cut off between the circle and the sides of the triangle.

Solution by MARGARET T. MEYER; D. BIDDLE; *and others.*



On the given base AB construct a segment of a circle containing an angle equal to the given vertical angle; and let QR, parallel to AB, be the given chord. Produce the chords QA, RB, to meet in O; divide AB in the given ratio at S, and through S draw OC, cutting the given chord in I, and the circumference in C. Then ABC will be the required triangle. For, let AC, CB meet QR in H and K; then, since QR is parallel to AB, we have

$$AS : SB = QI : IR = HI : IK = QH : KR.$$

7536. (By Professor SYLVESTER, F.R.S.)—If $3n-2$ points are given on a cubic curve, and through $3n-3\nu-2$ of these an $(n-\nu)$-ic be drawn, cutting the cubic in two additional points, and through these and the remaining 3ν given points a third curve of order $\nu+1$ be drawn, prove that its remaining intersection with the given cubic is a fixed point.

Solution by W. J. C. SHARP, M.A.

This theorem is a consequence of Professor SYLVESTER's theory of Residuation (*Reprint*, Vol. 34, p. 34). Taking any 3ν points, if A and B be the additional intersections of an $(n-\nu)$-ic through the other $3n-3\nu-2$ points, and C that of the $(\nu+1)$-ic, is through the 3ν points and A and B, the $3n-3\nu-2$ other points are coresidual to C and the original 3ν points. If A′, B′, C′ be corresponding points obtained by taking another $(n-\nu)$-ic through the same $3n-3\nu-2$ points, the 3ν points and C are coresidual to the same 3ν points and C′, and C and C′ denote the same point. This point is the single point coresidual to the original $3n-2$ points, for the $(n+1)$-ic system composed of the first $(n-\nu)$-ic and the corresponding $(\nu+1)$-ic is such that the $3n-2$ points are residual to A, A, B, B, and C. Also the n-ic through the $3n-2$ points and C will meet the cubic in a residual point P, which is, therefore, coresidual to A, A, B, B, and therefore residual to C, C (*i.e.* it is the tangential of C); and the $3n-2$ points are residual to P and C, and therefore coresidual to C, which is therefore the same, however the $3n-2$ points may be taken.

The theorem alluded to by Professor SYLVESTER is given in SALMON's *Higher Curves*, Art. 154, p. 131, and is shown to be a consequence of Professor SYLVESTER's theory of Residuation, Art. 160, p. 135; it is identical with Mr. J. J. WALKER's Quest. 7058.

If $n=3$ and $\nu=0$, the theorem becomes:—"If cubic curves be drawn through seven points on a given cubic, the lines joining the two remaining intersections of any of these with the original cubic will all pass through a fixed point upon it."

If $n=2$ and $\nu=1$, it becomes:—"If four points be given on a cubic and through any one of these a straight line be drawn meeting the cubic in two other points, the conic through these and the other three original points meets the cubic again in a fixed point; and, as a particular case of this, 'If a conic osculate a given cubic at a given point A and touch it at B, it will pass through the single point coresidual to the tangential of B, and three coincident points at A.'" In this way innumerable theorems may be deduced.

7512. (By Professor TOWNSEND, F.R.S.)—An ellipsoid and any inscribed polyhedron of maximum volume, or circumscribed polyhedron of minimum volume, being supposed to bound two solids of uniform density in their common space; show that both solids have the same principal axes at their common centre of inertia.

Solution by the PROPOSER; C. GRAHAM, M.A.; *and others.*

The polyhedron, whether inscribed or circumscribed, being always regular in the particular case of a sphere, therefore, for both solids, $\Sigma(yz\,dm)=0$, $\Sigma(zx\,dm)=0$, $\Sigma(xy\,dm)=0$, for every triad of rectangular planes passing through the common centre, in that particular case. And every triad of such planes for any mass or system of masses, transforming into a triad for which $\Sigma(y'z'\,dm) = 0$, $\Sigma(z'x'\,dm) = 0$, $\Sigma(x'y'\,dm) = 0$, in every transformation for which $x' = \lambda x$, $y' = \mu y$, $z' = \nu z$, where λ, μ, ν are constants; therefore, &c., for the general case.

7230. (By the Editor.)—On a square (A) of a chess-board, a knight is placed at random: find the probability that it can march (1) from that square (A) to a given square (B), as, for example, to one of the corner-squares, within a moves; and (2) over b squares in less than c moves, for instance, over the four corner-squares of the board.

Solution by D. Biddle.

In solving this problem, it is well to remember three things:—(*a*) That, according to his position, the knight's command of squares varies. When on one of the 16 central squares, he commands 8 squares; when on the 16 which flank the borders of the 4^2 central set, he commands 6 squares; when at the corners of the 6^2 set, he commands 4 squares, and when on any of the 4 middle squares of each side also, he commands 4 squares; when on a border square adjoining the corner on either side, he commands 3 squares; and when on a corner-square, he commands only 2. Consequently there are—

16	squares on which his range is	8,
16	,,	,,	6,
20	,,	,,	4,
8	,,	,,	3,
4	,,	,,	2,

and his average range is $5\frac{1}{4}$.

(*b*) That, as he moves from his original square 1, 2, 3 moves, his range undergoes a branching process; and his command of squares from one position overlaps that from another, being often partially similar, though never entirely the same. Thus, from either of the 4 central squares there are 8 once removed, but, instead of 8 times 8 twice removed, only 26.

(*c*) That the chess-board is so far symmetrical that an examination of the knight's progress from 10 out of the 64 squares is sufficient to give us data for the whole board. The 10 squares referred to are those which, roughly speaking, are included within the right-angled triangle whose hypotenuse is half a diagonal of the board starting from either corner. Each of those on the half-diagonal represents 4 similar ones, including itself; each of the other 6 represents 8 similar ones, including itself. Or, dividing the board into quarter blocks of 16 squares, E = B, I = C, N = D, K = G, O = H, and P = M. The only squares that have none corresponding in the same block are A, F, L, Q.

A	E	I	N
B	F	K	O
C	G	L	P
D	H	M	Q

The following table gives a precise account of the number of squares that can be reached by the knight from each of the 16 in 1, 2, 3, &c. moves:—

Original Square.	Reached in 1 move.	Reached in 2 moves.	Reached in 3 moves.	Reached in 4 moves.	Reached in 5 moves.	Reached in 6 moves.
A	8	26	24	5		
B	8	22	24	9		
C	6	20	26	11		
D	4	16	24	15	4	
E (=B)	8	22	24	9		
F	8	19	24	12		
G	6	17	25	14	1	
H	4	14	23	17	5	
I (=C)	6	20	26	11		
K (=G)	6	17	25	14	1	
L	4	13	26	18	2	
M	3	12	23	19	6	
N (=D)	4	16	24	15	4	
O (=H)	4	14	23	17	5	
P (=M)	3	12	23	19	6	
Q	2	9	20	21	10	1
Sum Totals	84	269	384	226	44	1
Average	$5\frac{1}{4}$ out of 63	$16\frac{13}{16}$ out of 63	24 out of 63	$14\frac{1}{8}$ out of 63	$2\frac{3}{4}$ out of 63	$\frac{1}{16}$ out of 63

1. From the foregoing table we find that the probability of (A) and (B), *both* taken at random, being within one move is $\frac{84}{1008} = \frac{1}{12}$; within 2 moves, $\frac{353}{1008}$; within 3 moves, $\frac{737}{1008}$; within 4 moves, $\frac{963}{1008}$; and within

5 moves, $\frac{1007}{1008}$. The opposite extremities of either diagonal are the only positions which take the knight *six* moves to march between. The chance of his being placed on a corner-square is $\frac{4}{64} = \frac{1}{16}$; and the chance of his having to march to the opposite corner, $\frac{1}{63}$. Consequently, $\frac{1}{16 \cdot 63} = \frac{1}{1008}$ is the probability of his having to take 6 moves in marching from one position to another, and this is the remainder left by $1 - \frac{1007}{1008}$ already found.

But, in the question as stated, (B) is *given*, and (A) alone taken at random. Moreover, one of the corner-squares is specially selected for (B). Now in our table Q is the corner-square, and we are able to state that if $a = 1$ move, the probability is $\frac{2}{63}$; if $a = 2$ moves, $\frac{11}{63}$; if $a = 3$ moves, $\frac{31}{63}$; if $a = 4$ moves, $\frac{52}{63}$; if $a = 5$ moves, $\frac{62}{63}$; and if $a = 6$ moves, then $\frac{63}{63} = 1$, or certainty. The sum of the figures in the table, opposite the letter denoting the square selected for (B) (according to the question), up to and including those in the column devoted to the number of moves selected for a, will always be the numerator, and 63 always the denominator, of the probability required.

2. The second part of the question is more complicated, since it is impossible to tell which of the b squares may be nearest to (A); and the number of moves will vary not only according to the distance of (A) from the series b, but also according to the order in which the b squares are taken, unless, as in the instance given in the question, they are symmetrically placed. We can, however, discover the average moves taken by the knight in crossing from one position to another, both taken at random, by multiplying the sum totals given in the table by the numbers given at the head of the several columns; then taking the sum, and dividing by 1008 (their original grand total). Thus

$$\frac{1}{1008}(84 + 538 + 1152 + 904 + 220 + 6) = 2{\cdot}881 \text{ nearly,}$$

or $\frac{2904}{1008}$. And this multiplied by b will give the average number of moves taken by the knight in marching from (A) to the series b and through it; because, though $b + 1$ = the number of squares marking the several positions of the knight from (A) onwards, yet the number of intervals $= b$ only. Of course, c must always equal or exceed b.

In the instance given, of the 4 corner-squares, we know that c must equal or exceed 15, to come within the range of probability, because 5 is the lowest number of moves taken by the knight in moving from one corner to another; and we will even allow that (A) may be itself a corner-square. In any case, the distance of (A) from a corner-square never exceeds 3 moves. Consequently, if c be 18 or more, the probability = 1 or certainty; and if c be 14 or under, the probability = 0. There are 16 squares from which the knight can reach one or other corner in 3 moves; 36 from which he takes 2 moves; 8 from which he takes 1 move; and 4 which are corner-squares, and in which the move must be reckoned 0. Therefore, if $c = 17$, the probability $= \frac{3}{4}$, that is $\frac{4 + 8 + 36}{64}$; if $c = 16$, the probability $= \frac{3}{16}$, that is $\frac{4 + 8}{64}$; and if $c = 15$, the probability $= \frac{1}{16}$, or $\frac{4}{64}$. Of course, if (A) cannot under the circumstances be a corner-square, the probabilities for 17, 16, and 15 will be $\frac{11}{15}$, $\frac{2}{15}$, and 0 respectively.

7622. (By SYAMA CHARAN BASU, B.A.)—PSQ is a focal chord of a parabola; tangents PR, QR intersect in R. Show that the third tangent parallel to PSQ bisects RS at right angles.

Solution by CHRISTINE LADD FRANKLIN, M.A.; KATE GALE; *and others.*

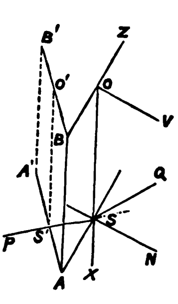
Any line parallel to PQ is at right angles to RS. That the parallel tangent bisects RS is a reciprocal of the proposition that a chord of a circle is bisected by the diameter perpendicular to it. Thus :—

A line through any point on a circle cuts the circle again in a point which is fourth harmonic to the first point, the point at infinity, and the intersection of the line with the diameter perpendicular to it.

The tangent parallel to a focal chord passes through the fourth harmonic to the point at infinity, the pole of that chord, and the focus.

6053. (By the Rev. A. J. C. ALLEN, B.A.)—A prism filled with fluid is placed with its edge vertical, and a beam of light is passed through an infinitely thin vertical slit, and is incident normally on the prism infinitely near its edge. The emergent beam is received on a vertical screen. If the refractive index of the fluid varies as the depth below a horizontal plane, find the nature and position of the bright curve formed in the screen.

Solution by J. J. WALKER, M.A.



Let AB be the edge of the prism, ABB'A' the face on which a ray of the beam, as PS'S, is incident (at the point S') perpendicularly, emerging from the other face ABOS at the point S. The points of incidence lying on a line S'O' parallel to the edge AB, and consequently the points of emergence in another parallel SO, it is immaterial whether these points be supposed infinitely near the edge or not. Let SQ be the emergent ray, SN being the normal to the second face of the prism. Let YOZ be the horizontal plane to the distance of S from which the refractive index at S is proportional, OY being normal to and OZ lying on the face of the prism. Then, taking OSX, OY, OZ as axes, the equations to

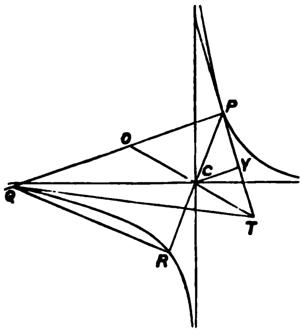
SQ are ($OS = x'$) $\quad x = x',\ y \sin QSN = z \cos QSN,$

and ($\angle SAS' = \alpha$) $\quad \sin QSN = k'x' \sin\alpha = x'/k,$ suppose;

so that, eliminating x' and $\angle QSN$, there results $y^2x^2 = z^2(k^2 - x^2)$, as the equation of the surface generated by the emergent rays, and the bright curve formed on the vertical screen $y = mz + n$ will be a quartic curve.

7543. (By Professor WOLSTENHOLME, M.A., Sc.D.)—In a rectangular hyperbola, PQ is a chord normal at P, and T is its pole: prove that CT will be at right angles to CP; that is, T is the extremity of the polar subtangent drawn from the centre C. [*Otherwise:* if O be the mid-point of PQ, the angle OCP will be a right angle.]

Solution by (1) J. A. OWEN, B.Sc.; (2) MARGARET T. MEYER.



1. By a known theorem, PQ is equal to the diameter of curvature at P, or

$$PO = \text{radius of curvature at P} = CP^2 / CY;$$

that is, $OP : PC = PC : CY$,

and the angles OPC, PCY are equal; hence the triangles OPC, PCY are similar, and the angle PCO is therefore a right angle; and also the angle PCT.

2. Let TC meet PQ in O, and PC meet the curve again in R; then $QO = PO$, $RC = CP$, hence TO is parallel to QR. But, since the hyperbola is rectangular, the angle QRP subtended by PQ at the other extremity of PCR, the diameter through P is equal to the angle between PQ and the tangent at P, that is, since PQ is a normal chord, QR is perpendicular to CP; hence CT is at right angles to CP, or OCP is a right angle.

7545. (By J. J. WALKER, M.A., F.R.S.)—Prove that the points on a right line have a (1, 1) correspondence with the rays of a pencil in the same plane; show that the lines drawn from the points so as to make a given angle with their corresponding rays all touch a parabola, which is also touched by the given right line. [A generalisation of a theorem of STEINER'S.]

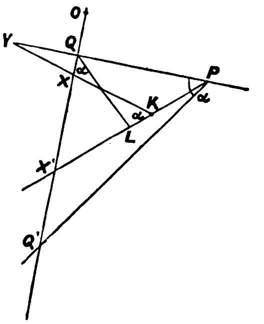
Solutions by (1) Prof. WOLSTENHOLME, Sc.D.; (2) *the* PROPOSER.

1. If the ray corresponding to any point X on the right line meet the right line in X′, the points X, X′ will have a (1, 1) correspondence, and there will be a definite point O on the right line such that $OX\,(OX'+h)$ will be constant (c^2), h being a definite constant. Take O for origin and the right line for axis of x; and let (a, b) be the vertex of the pencil; then, if $\dfrac{x-a}{\cos\theta} = \dfrac{y-b}{\sin\theta}$ be any ray, and α the given angle, the straight line drawn through the point corresponding

to this ray will have for equation $y = \tan(\theta + \alpha)\left(x - \dfrac{c^2}{a + h - b\cot\theta}\right)$, which involves $\tan\theta$ in the second degree, so that the envelope is a conic; and, since the straight line is altogether at infinity when $a + h = b\cot\theta$, this conic must be a parabola.

The following further generalisation is obvious:—

If the points on a right line have a (1, 1) correspondence with the rays of a pencil, the straight line drawn through any point on the right line so as with its corresponding ray to divide a given segment in a given cross ratio, will envelope a conic, touching the given segment and the given right line.



2. The question may be solved more in the Steinerian manner as follows:—Let Q, Q′ be corresponding points on the right line which subtend at P, the vertex of the pencil, an angle equal to the given angle (α); X, X′ being any other corresponding points: viz., let PQ′, PX′ be the rays corresponding to Q; X. Let XK, making with PX′ the given angle (α), meet PQ in Y; and QL, meeting PX′ in L, make with the given right line the same angle (α). Then we have

$$\begin{aligned} QY : QX &= \sin X : \sin Y \\ &= (PL + LX')\sin L : PX' \sin Q'PX' \\ &= PQ \sin PQL + QX' \sin\alpha : X'Q' \sin Q' \\ &= PQ \sin PQL - OQ \sin\alpha + OX' \sin\alpha : OQ' \sin Q' - OX' \sin Q'; \end{aligned}$$

O being the point on the given range corresponding to the parallel-ray, and PO′ the ray corresponding to the point of the range at infinity; for which $OX'.O'X = c^2 = OQ'.O'Q$. Hence, multiplying both terms of the latter ratio by O′X, O′Q, and substituting, there results

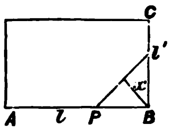
$$QY : QX = h . O'X + O'Q \sin\alpha . QX \sin Q',$$

where $c^2h = O'Q\,(PQ \sin PQL - OQ \sin\alpha)$, or $QY \sin Q' = hO'X + O'Q \sin\alpha$, or, if O″ is a point on PQ such that $QO'' \sin Q' = O'Q \sin\alpha$, then $O''Y : O'X$ is a constant ratio; and consequently XY touches a given parabola, to which O′X and O″Y are also tangents.

[The parabolic envelope manifestly touches the given straight line, and its axis makes with the straight line that joins O to the vertex of the pencil an angle equal to the given angle. Christine Ladd Franklin remarks that, if we revolve the pencil through an angle equal to the given angle, the construction becomes the ordinary projective construction for a parabola,—which is, in fact, one of the two cases given by Steiner, viz., when the lines are drawn parallel to the rays of the pencil or perpendicular,—but he does not seem to have seen his way to the general case, though why it is difficult to conjecture.]

7611. (By B. Reynolds, M.A.)—A man, having to pass round the corner of a rectangular ploughed field, strikes across the field diagonally, at 45°, upon nearing the corner, to save time. If his velocity on the beaten path is u, and that on the field is $u-x$, where x is the perpendicular distance of the path chosen from the corner, find (1) where he should leave the beaten path, and (2) what value of x will make either route occupy the same time.

Solution by W. G. Lax, B.A.; C. Morgan, B.A.; *and others.*



1. Let l and l' be the lengths of the sides of the field adjacent to the angle the man cuts across; then

time from A to C = $\dfrac{l+l'-2\sqrt{2}\,.\,x}{u}+\dfrac{2x}{u-x}$,

and, for this to be a minimum, we have

$$\frac{-2\sqrt{2}}{u}+\frac{2(u-x)+2x}{(u-x)^2}=0, \text{ whence } x=u\left(1-\frac{1}{\sqrt[4]{2}}\right).$$

2. Time round corner B = $\dfrac{l+l'}{u}$, and, for this to be same as across,

$$\frac{l+l'-2\sqrt{2}x}{u}+\frac{2x}{u-x}=\frac{l+l'}{u},\quad \frac{2\sqrt{2}x}{u}-\frac{2x}{u-x}=0,\quad \text{and } x=u\left(1-\frac{1}{\sqrt{2}}\right).$$

[The Proposer remarks that he is "afraid that the assumed law of diminution of velocity is a very unscientific one, especially because u involves *time*, and x does not. If we make u = 120 *yards* per *minute*, the law seems reasonable, but with the same velocity, denoted as 2 yards per second, the law seems ridiculous. It seems also a pity for the diminished velocity $(u-x)$ to have a possibility of becoming zero or negative."]

7391. (By the Editor.)—Find the area of an inscriptible quadrilateral whose sides are roots of the equation $x^4+px^3+qx^2+rx+k=0$, and deduce therefrom a solution of Quest. 7330 (*Reprint*, Vol. 39, p. 111).

Solution by Dr. Curtis; S. Greenidge, M.A.; *and others.*

If $s=\frac{1}{2}(a+b+c+d)=-\frac{1}{2}p$, we have

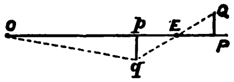
$$\begin{aligned}(\text{Area})^2 &= (s-a)(s-b)(s-c)(s-d),\\ &= s^4-(a+b+c+d)\,s^3+(ab+ac+\&c.)\,s^2-(abc+\&c.)\,s+abcd\\ &= -s^4+qs^2+rs+k=-\frac{p^4}{2^4}+\frac{p^2}{2^2}q-\frac{pr}{2}+k.\end{aligned}$$

When $k=0$, one side of the quadrilateral vanishes, the quadrilateral degenerates into a triangle, and we obtain the result in Quest. 7330.

7601. (By Professor Hudson, M.A.)—The lenses of a common astronomical telescope, whose magnifying power is 16, and length from object-glass to eye-glass $8\frac{1}{2}$ inches, are arranged as a microscope to view an object placed $\frac{5}{8}$ of an inch from the object-glass; find the magnifying power, the least distance of distinct vision being taken to be 8 inches.

Solution by B. H. Rau, M.A.; Sarah Marks; *and others.*

Let F and f be the focal lengths of the object- and eye-glasses of the telescope. Then, by question, $F+f = 8\frac{1}{2}$ in., and $\frac{F}{f} = 16$; therefore $F = 8$ in., $f = \frac{1}{2}$ in. When arranged as a microscope, these glasses are interchanged in order.



Let O, E, PQ, pq, be the centres of the object-glass, eye-glass (of the telescope), the object viewed, and its image, respectively.

Then $Op = F = 8$ in.; $PE = \frac{5}{8}$ in.,

also $\frac{1}{EP}+\frac{1}{Ep} = \frac{1}{f}$; whence $\frac{Ep}{EP} = \frac{\frac{1}{2}}{\frac{5}{8}-\frac{1}{2}} = 4$. Now the magnifying

power $= \frac{pq}{Op} \div \frac{PQ}{\text{distance of distinct vision}} = \frac{pq}{Op}\cdot\frac{8}{PQ} = \frac{8}{8}\cdot\frac{pE}{PE} = 4.$

7605. (By J. J. Walker, M.A., F.R.S.)—Referring to Question 1585, show that (1) the circles drawn on the common chords of three mutually orthotomic circles as diameters have not a common radical axis (as erroneously stated in that Question), but have the same radical centre as those circles; and (2) their common chords are equal to one another, and (3) respectively parallel to the radii of the circle through the centres of the orthotomic triad, drawn to those centres.

Solution by Asûtosh Mukhopâdhyây.

My solution to Quest. 1585, is incorrect; the x-coordinate of C being in reality $-b\cos A\cos B$, so that the first term of equation (3) should be $(x+b\cos A\cos B)^2$. Equation (4) is correct. But, subtracting (3) *as corrected* from (1), we get, not the same equation as (4), but

$$-x\cos A + y\sin A = \tfrac{1}{2}(c\cos C + b\cos B - a\cos A)\;\ldots\ldots\ldots\ldots(5).$$

Hence, the circles have *not* the same radical axis. Solving for x and y from (4) and (5), we get for the coordinates of the radical centre of the circles (1), (2), (3), $x = 0$, $y = \frac{1}{2}\operatorname{cosec} A\,(c\cos C + b\cos B - a\cos A)$.

But these are well known to be the coordinates of O. Hence, the three circles on the common chords of the orthotomic triad as diameters, *instead of having a common radical axis*, have the same radical centre as the three original mutually orthotomic circles. [This can otherwise be proved by reasoning similar to what is followed in Townsend's *Modern Geometry*, Vol. i., Art. 183, Cor. 3.]

Again, if P be the circumcentre of the triangle ABC, its coordinates are easily seen to be $-\frac{1}{2}a + b\cos C,\ \frac{1}{2}a\cot A$; but C is $(2\cos C,\ 0)$; hence the line PC is

$$\frac{x - b\cos C}{y} = \frac{\frac{1}{2}a}{-\frac{1}{2}a\cot A}, \text{ or } x\cos A + y\sin A = b\cos C\cos A \ \ldots\ldots(6).$$

This is obviously parallel to (4), which is the radical axis of (1) and (2). Hence, we infer that the radical axes of the circles on the common chords of the mutually orthotomic circles as diameters are parallel to the radii of the circle passing through the centres of the orthogonal triad, drawn to these centres.

Again, we see that (4) and (5) differ only in this, that the intercepts they make on the axis of x are on opposite sides of the origin; hence, it follows, from elementary geometry, that the portions of these lines which form chords of (1) are equal,—as, indeed, can be shown by direct calculation, since $(\text{chord})^2 = 4\,bc\cos B\cos C - (c\cos C + b\cos B - a\cos A)^2$, in both cases; this interprets that the common chords of the three circles are equal.

7593. (By R. KNOWLES, B.A., L.C.P.)—A circle passes through the ends of a chord PQ of the parabola $y^2 = 4ax$ and its pole (hk); prove that (1) its equation is $x^2 + y^2 - \frac{k^2 - 2a^2}{a}x - \frac{k}{a}(a-h)\,y + h\,(2a-h) = 0$; (2) if PQ is perpendicular to the axis, the focus is the centre; (3) if the circle cuts the parabola again in CD, the middle point of the line joining the poles of PQ and CD, with respect to the parabola, is the focus.

Solution by MARGARET T. MEYER; B. H. RAU, M.A.; *and others.*

The equation to PQ, the polar of (h, k) with respect to the parabola $y^2 = 4ax$ is $ky - 2ax - 2ah = 0$. The equation to CD must be of the form $ky + 2ax + l = 0$ (since PQ and CD are equally inclined to the axis of the parabola), and that to any conic through P, Q, C, D is

$$y^2 - 4ax + \lambda\,(ky - 2ax - 2ah)(ky + 2ax + l) = 0,$$

and if this represents a circle, we have

$$(1 + \lambda k^2) + \lambda\,.\,4a^2 = 0, \text{ whence } \lambda = -\,(k^2 + 4a^2)^{-1};$$

hence the equation to the circle is

$$(k^2 + 4a^2)(y^2 - 4ax) - k^2y^2 + 4a^2x^2 + 2ah\,(ky + 2ax) - l\,(ky - 2ax - 2ah) = 0,$$

and $l = 2a\,(2a - h)$, because the circle passes through (hk). Thus the circle becomes $x^2 + y^2 - \frac{k^2 + 2a^2}{a}x - \frac{k}{a}(a-h)\,y + h\,(2a-h) = 0$. If PQ is perpendicular to the axis, $k = 0$, and the centre is the point $(a,\ 0)$, *i.e.* the focus. The pole of CD, *i.e.* $ky + 2ax + 2a\,(2a-h) = 0$, is the point $(2a-h,\ -k)$. The coordinates of the mid-point of the line joining the poles of PQ and CD with respect to the parabola are $\frac{1}{2}(h + 2a - h)$ and $\frac{1}{2}(k - k)$, *i.e.* $(a,\ 0)$, *i.e.* those of the focus.

7294. (By A. McMurchy, B.A.)—Without knowing the angles of a triangular prism, show that its refractive index can be determined by observing the minimum deviations of rays passing in the neighbourhood of the three angles; and if these deviations be denoted by 2α, 2β, 2γ, then μ is given by

$$\mu^3-\mu^2(\cos\alpha+\cos\beta+\cos\gamma)$$
$$+\mu[\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)]-\cos(\alpha+\beta+\gamma)=0.$$

Solution by D. Edwardes; Professor Nash, M.A.; *and others.*

If θ, ϕ, ψ be the angles of the prism, and 2α the minimum deviation at one angle, then it is known that $\sin(\alpha+\frac{1}{2}\theta)=\mu\sin\frac{1}{2}\theta$, and similarly for 2β and 2γ. Also, since $\theta+\phi+\psi=180°$,

$$\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi+\tan\tfrac{1}{2}\phi\tan\tfrac{1}{2}\psi+\tan\tfrac{1}{2}\psi\tan\tfrac{1}{2}\theta=1,$$

and $\tan\frac{1}{2}\theta=\dfrac{\sin\alpha}{\mu-\cos\alpha}$, &c. &c., whence, substituting and reducing, we have the result in the question.

7372. (By R. Russell, B.A.)—Determine $\theta(x)$ and $\phi(x)$ where they are of the form $\dfrac{Ax+B}{Cx+D}$, so that, by putting $y=\theta(x)$ or $\phi(x)$, the quartic $(abcde)(x,1)^4=0$ and its Hessian may turn into the quartic $(abcde)(y,1)^4=0$ and its Hessian.—(*a*) The determination of $\theta(x)$ and $\phi(x)$ depends on the solution of a cubic. (*b*) The roots of the quartic may be represented in the form α, $\theta(\alpha)$, $\phi(\alpha)$, $\theta[\phi(\alpha)]$.

Solution by the Proposer; Professor Matz, M.A.; *and others.*

If X, Y, Z be the quadratic factors of the G-covariant of the quartic, we know that $X^2+Y^2+Z^2\equiv 0$, $\lambda X^2+\mu Y^2+\nu Z^2=U$, and that X, Y, Z are connected harmonically in pairs. Hence (by the Question I have proposed), if we transform $(abcde)(x,y)^4$ by any of the substitutions,

$$\left.\begin{aligned}\xi&=\frac{dX}{dy}\\ \eta&=-\frac{dX}{cx}\end{aligned}\right\}(1),\qquad \left.\begin{aligned}\xi&=\frac{dY}{dy}\\ \eta&=-\frac{dY}{cx}\end{aligned}\right\}(2),\qquad \left.\begin{aligned}\xi&=\frac{dZ}{dy}\\ \eta&=-\frac{dZ}{cx}\end{aligned}\right\}(3),$$

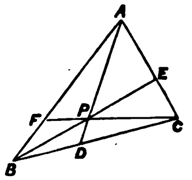
the quadratics X, Y, Z retain exactly their forms, and therefore the transformed quartic is $(abcde)(\xi,\eta)^4$.

7594. (By W. J. C. Sharp, M.A.)—If the circle inscribed in the triangle ABC touch the sides at the points D, E, F respectively, and P be the point of concurrence of the lines AD, BE, CF; and again, if D′, E′, F′, P′ be the corresponding points for the escribed circle opposite A, show that $\frac{PD}{AD}+\frac{PE}{BE}+\frac{PF}{CF}=1, \quad -\frac{P'D'}{AD'}+\frac{P'E'}{BE'}+\frac{P'F'}{CF'}=1$ ...... (1, 2).

[In the second result, the lines are considered as signless magnitudes; if regard were had to the signs, the − should be omitted.]

Solution by Belle Easton; B. H. Rau, M.A.; *and others.*

1 This is true of any three straight lines AD, BE, CF passing through a common point P.



For $\frac{PD}{AD}=\frac{\Delta BPD}{\Delta BAD}=\frac{\Delta CPD}{\Delta CAD}=\frac{\Delta BPC}{\Delta ABC},$

$$\frac{PE}{BE}=\&c., \quad \frac{PF}{CF}=\&c.;$$

hence

$$\frac{PD}{AD}+\frac{PE}{BE}+\frac{PF}{CF}=\frac{\Delta BPC+\Delta CPA+\Delta APB}{\Delta ABC}=\frac{\Delta ABC}{\Delta ABC}=1.$$

2. Since P′ is on the side of BC remote from A,

$$-\Delta PBC+\Delta PCA+\Delta PBA=\Delta ABC;$$

hence we have $-\frac{P'D}{AD}+\frac{P'E}{BE}+\frac{P'F}{CF}=1.$

7541. (By Professor Wolstenholme, M.A., Sc.D.)—The coordinates of a point being $x=a(m^2+m^{-2})$, $y=a(m-m^{-1})$, where m is the parameter, according to the usual rule the locus should be a quartic, since we get four values of m for determining the points in which the locus meets any proposed straight line. Nevertheless, the locus is the parabola $y^2=a(x-2a)$. Account for the discrepancy. Also, with the same values of (x, y), the equation of the tangent is $m^2x-2m(m^2-1)y+a(m^4-4m^2+1)=0$, which would make the class number 4.

Solution by W. J. C. Sharp, M.A.; R. Knowles, B.A.; *and others.*

The explanation is that $m^2+m^{-2}=(m-m^{-1})^2+2=t^2+2$, suppose, and so the equations may be written $x=a(t^2+2)$, $y=at$, which of course represents a unicursal curve of the second order; also the equation to the tangent may be put in the form $x-2(m-m^{-1})y+a(m^2-4+m^{-2})=0$ or $x-2ty+a(t^2-2)=0$, so that the curve is of the second class. If α be the inclination of the tangent at (x, y) to the axis, $m=-\tan\frac{1}{2}\alpha$ or $\cot\frac{1}{2}\alpha$.

[The proper resultant of the two equations is $(y^2-ax+2a^2)^2=0$, so that the parabola should be considered as doubled, every straight line meeting it in two pairs of coincident points; and every tangent counting as 2. Every point on the curve and every tangent is given twice (for the values k, $-k^{-1}$ of m): this implies that every point of the locus is a node and every tangent a bitangent, which can only happen when a curve is double.

Of course the change of notation adopted above obviates all difficulty. Hence, writing down both resultants as determinants, it follows that

$$\begin{vmatrix} a, & 0, & -x, & 0, & a, & 0, & 0 \\ 0, & a, & 0, & -x, & 0, & a, & 0 \\ 0, & 0, & a, & 0, & -x, & 0, & a \\ a, & -y, & -a, & 0, & 0, & 0, & 0 \\ 0, & a, & -y, & -a, & 0, & 0, & 0 \\ 0, & 0, & a, & -y, & -a, & 0, & 0 \\ 0, & 0, & 0, & a, & -y, & -a, & 0 \\ 0, & 0, & 0, & 0, & a, & -y, & -a \end{vmatrix} = \begin{vmatrix} a, & 0, & 2a-x, & 0 \\ 0, & a, & 0, & 2a-x \\ 0, & a, & -y, & 0 \\ 0, & 0, & a, & -y \end{vmatrix}^2 = a^4 (y^2 - ax + 2a^2)^2].$$

7492. (By W. J. C. Sharp, M.A.)—Show that at an inflexion on the curve $U = 0$, $\begin{vmatrix} u_{11}, & u_{12}, & u_1 \\ u_{12}, & u_{22}, & u_2 \\ u_1, & u_2, & 0 \end{vmatrix} = 0$. [This is an application of the form of the Hessian suggested at the end of the Solution of Question 5762.]

Solution by G. B. Mathews, B.A.; J. O'Regan; *and others.*

At an inflexion we have by Euler's theorem,

$$0 = \begin{vmatrix} u_{11}, & u_{12}, & u_{13} \\ u_{21}, & u_{22}, & u_{23} \\ u_{31}, & u_{32}, & u_{33} \end{vmatrix} = \begin{vmatrix} u_{11}, & u_{12}, & xu_{11}+yu_{12}+zu_{13} \\ u_{21}, & u_{22}, & xu_{21}+yu_{22}+zu_{23} \\ u_{31}, & u_{32}, & xu_{31}+yu_{32}+zu_{33} \end{vmatrix} = \begin{vmatrix} u_{11}, & u_{12}, & (n-1)\,u_1 \\ u_{21}, & u_{22}, & (n-1)\,u_2 \\ u_{31}, & u_{32}, & (n-1)\,u_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_{11}, & u_{12}, & (n-1)\,u_1 \\ u_{21}, & u_{22}, & (n-1)\,u_2 \\ (n-1)\,u_1, & (n-1)\,u_2, & (n-1)\,[xu_1+yu_2+zu_3] \end{vmatrix}$$

similarly $$= \begin{vmatrix} u_{11}, & u_{12}, & (n-1)\,u_1 \\ u_{21}, & u_{22}, & (n-1)\,u_2 \\ (n-1)\,u_1, & (n-1)\,u_2, & n\,(n-1)\,u \end{vmatrix};$$

or, since $u = 0$ at all points on the curve, the stated result follows.

7523. (By S. Tebay, B.A.)—Find the mean value of the radius of curvature for all points of an ellipse.

Solution by B. H. Rau, M.A.; A. Mukhopâdhyây; *and others.*

The radius of curvature at the point xy of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is $\rho = \frac{a^2}{b}\left(1-e^2\frac{x^2}{a^2}\right)^{\frac{3}{2}}$. Let ϕ be the complement of the eccentric angle of the point xy; then $x = a\sin\phi$, and $y = a\cos\phi$; therefore

$$\rho = \frac{a^2}{b}\,(1-e^2\sin^2\phi)^{\frac{3}{2}}.$$

The mean value of ρ is $\dfrac{\int \rho\, ds}{\int ds}$; and $\dfrac{ds}{d\phi} = a\,(1-e^2\sin^2\phi)^{\frac{1}{2}}$;

$$\therefore \quad s = 4a\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\phi)^{\frac{1}{2}}\,d\phi = 2\pi a\left(1-\tfrac{1}{4}e^2-\tfrac{3}{64}e^4-\tfrac{5}{256}e^6\ldots\right),$$

$$\int \rho\, ds = 4\,\frac{a^3}{b}\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\phi)^2\,d\phi = \frac{2\pi a^3}{b}\left(1-e^2+\tfrac{3}{8}e^4\right).$$

$$\therefore \quad \rho' = \frac{a^2}{b}\cdot\frac{1-e^2+\frac{3}{8}e^4}{1-\frac{1}{4}e^2-\frac{3}{64}e^4-\frac{5}{256}e^6-\ldots} = \frac{a^2}{b}\left(1-\tfrac{3}{4}e^2+\tfrac{15}{64}e^4+\tfrac{11}{256}e^6+\ldots\right).$$

7587. (By Syama Charan Basu, B.A.)—If

$$\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)\left(\frac{b}{c}+\frac{c}{b}\right)+4 = 0,$$

where α, β are the roots of $ax^2+bx^2+c = 0$, show that $\alpha = \beta = 2$.

Solution by Margaret T. Meyer ; B. H. Rau, M.A. ; *and others.*

Since $\dfrac{b}{c} = -\dfrac{\alpha+\beta}{\alpha\beta}$, therefore we have

$$\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)\left(\frac{\alpha+\beta}{\alpha\beta}+\frac{\alpha\beta}{\alpha+\beta}\right) = 4,\quad \left(\frac{\alpha+\beta}{\beta^2}+\frac{\beta^2}{\alpha+\beta}-2\right)+\left(\frac{\alpha+\beta}{\alpha^2}+\frac{\alpha^2}{\alpha+\beta}-2\right) = 0,$$

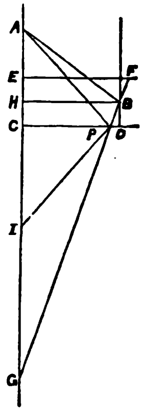
$$\left\{\frac{(\alpha+\beta)^{\frac{1}{2}}}{\beta}-\frac{\beta}{(\alpha+\beta)^{\frac{1}{2}}}\right\}^2+\left\{\frac{(\alpha+\beta)^{\frac{1}{2}}}{\alpha}-\frac{\alpha}{(\alpha+\beta)^{\frac{1}{2}}}\right\}^2 = 0,\ \therefore\ \alpha+\beta = \beta^2 = \alpha^2,$$

and, since $\alpha+\beta$ is not zero, we have $\alpha = \beta = 2$.

7576. (By the Editor.)—Two houses (A, B) stand 750 yards apart on the side of a hill of uniform slope, and at the respective distances of AC = 600 yards and BD = 150 yards from a brook that runs in a straight line CD along the foot of the hill. A man starts from the house A to go to the brook for water, which he is to carry to the house B. Supposing he can only walk 2 miles an hour in going up hill with the water, but 4 miles an hour in going down hill to the brook: show that (1), in order to perform his work in the shortest possible time, he must strike the brook at a point P such that CP = 546·124 yards, the distance he will travel is AP + PB = 811·494 + 159·298 = 970·79 yards, and the time the walking part of his journey will take is 6·916 + 2·715 = 9·631 minutes; also (2), if he start from B to return likewise to A, he will have to take the water at the mid-point (Q) of CD, the length of his return journey will be $450\sqrt{5}$ = 1006·23 yards, the time will be $\frac{1125}{176}\sqrt{5}$ = 14·293 minutes, and the two parts BQ, QA of his path will be perpendicular to each other.

Solution by D. BIDDLE; W. J. BARTON, M.A.; *and others.*

1. This is equivalent to finding the shortest combination of straight lines from A to CD and through B to EF (parallel to CD and equidistant with it from B). And the position of P is identical with that of the point of equilibrium, when a body capable of free movement along CD (but in no other direction) is under the influence of forces operating from A and B, and when the force acting along AP is half that acting along PB. In other words, cos APC = 2 cos BPD; and if AC and BP be produced to meet in G, then PG = 2AP. According to *data*, it is readily seen that HB = AC.



Let BD (= 150 yards) be the unit of length, and put CP = x; then, since AC = 4, we have

$$(x^2+CG^2)^{\frac{1}{2}} = 2(x^2+16)^{\frac{1}{2}},$$

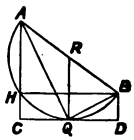
whence $CG^2 = 3x^2+64$; also CG : x = CG + 1 : 4, whence CG = $x \div (4-x)$; thus we have

$$(3x^2+64)(4-x)^2 = x^2,$$

or $$3x^4-24x^3+111x^2-512x+1024 = 0 \quad\ldots\ldots\ldots(\alpha).$$

There are two real roots of this equation, and the one required is $x = 3{\cdot}6408273$, whence the stated results follow.

2. Regarding the return journey in the particular instance, we can find the point Q by bisecting AB in R, and with R as centre, and RB as radius, drawing the semicircle AHQB, which will *touch* CD in Q, since RQ = $\frac{1}{2}$AB = BD + $\frac{1}{2}$AH = the shortest distance from R to CD. Thus RQ is perpendicular to CD and parallel to AC and BD; moreover ∠AQR = CAQ, ∠BQR = DBQ, and AQB is a right angle; also CQ = QD = 2BD = $\frac{1}{2}$AC = $\frac{1}{2}$CD; and cos BQD : cos AQC = cot BQD = 2, or cos BQD = 2 cos AQC, as required by the minimum condition; hence the stated results follow.



[Putting v, $2v$ for the respective velocities up and down hill, and θ, ϕ for the angles PAC, PBD, the conditions of part (1) of the problem are expressed by either of the two following sets of equations:—

$$4\tan\theta+\tan\phi = 4, \quad 2\sec\theta+\sec\phi = \text{minimum} \quad\ldots\ldots\ldots(\beta,\gamma),$$

$$(x^2+16)^{\frac{1}{2}}+2\left[(4-x)^2+1\right]^{\frac{1}{2}} = \text{minimum} \quad\ldots\ldots\ldots\ldots\ldots\ldots(\delta).$$

Differentiating (β), (γ), and dividing one result by the other, we get

$$\sin\theta = 2\sin\phi \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\epsilon).$$

Differentiating (δ), we have $\dfrac{x}{(x^2+16)^{\frac{1}{2}}} = \dfrac{2(4-x)}{(x^2-8x+17)^{\frac{1}{2}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots(\zeta).$

Of these equations, (ϵ) is equivalent to cos APC = 2 cos BPD, and (ζ) reduces to (α) above.

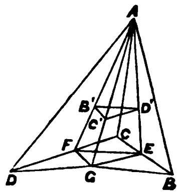
In part (2) of the problem, if DQ = x, the equation of condition is

$$(x^2+1)^{\frac{1}{2}}+2\left[(4-x)^2+16\right]^{\frac{1}{2}} = \text{minimum},$$

whence $\dfrac{x}{(x^2+1)^{\frac{1}{2}}} = \dfrac{2(4-x)}{\left[(4-x)^2+16\right]^{\frac{1}{2}}}$, which is satisfied by $x = 2$, thus length of path = BQ + QA = $3\sqrt{5}$ = (as unit is 150 yards) 1006·23 yds.; also $BQ^2+QA^2 = 25 = AB^2$, so that AQB is a right angle.]

7533. (By J. J. WALKER, M.A., F.R.S.)—Prove that the common centre of the surface-mass of the four faces of a tetrahedron is the centre of the sphere inscribed in that determined by the four centres of the faces; and hence prove the obvious analogue in tri-dimensional space of Professor HUDSON's Question 7488—which is true in any position of the point O, for forces proportional to OA sin A, OB sin B, OC sin C.

Solution by B. HANUMANTA RAU, M.A.; J. O'REGAN; *and others.*



Let A′, B′, C′, D′ be the centres of gravity of the faces opposite to the corners A, B, C, D of the tetrahedron ABCD, and let AB′, AC′, AD′ meet CD, DB, BC at their mid-points F, G, E; then, since

$$\frac{AC'}{AG} = \frac{AB'}{AF} = \frac{AD'}{AE} = \tfrac{2}{3},$$

therefore $\triangle B'C'D' = \tfrac{4}{9}\triangle GEF = \tfrac{1}{9}\triangle BCD.$

Hence the mass at A′ = 9△B′C′D′, and the distance of H, the common centre of surface mass of the faces, from the plane BCD is equal to

$$\frac{\text{mass at } A' \times \text{perpendicular from } A' \text{ on } B'C'D'}{\text{sum of the masses at } A', B', C', D'}$$

$$= \frac{27 \times \text{vol. of tetrahedron } A'B'C'D'}{9 \times \text{sum of the faces}},$$

a symmetrical result. Hence H is the centre of the sphere inscribed in the tetrahedron A′B′C′D′.

[As to the generalization of Question 7488, the PROPOSER remarks as follows:—Through any point O let forces act along the lines OA′... OD′, proportional to OA′ × B′C′D′... OD′ × A′B′C′, their resultant will act along the line drawn to the common centre (H) of masses placed at A′... D′ and equal to the surface-masses B′C′D′... A′B′C′ respectively; and this resultant will be proportional to OH × surface of A′B′C′D′. But this mass-centre has been shown to coincide with the centre of the sphere inscribed in A′B′C′D′.]

7556. (By W. NICHOLLS, B.A.)—Two cubics U and V have the same points of inflexion. Show that the intersection of the tangent at any point on U and the polar of that point with respect to V lies on U.

Solution by G. B. MATTHEWS, M.A.; SARAH MARKS; *and others.*

Let the cubics be $U \equiv x^3 + y^3 + z^3 + 6lxyz$, $V \equiv x^3 + y^3 + z^3 + 6mxyz$,

where (ξ, η, ζ) is any point on U; then the tangent is

$$(\xi^2 + 2l\eta\zeta)\,x + (\eta^2 + 2l\zeta\xi)\,y + (\zeta^2 + 2l\xi\eta)\,z = 0;$$

and the polar line of (ξ, η, ζ) with respect to V is

$$(\xi^2 + 2m\eta\zeta)\,x + \ldots + \ldots = 0;$$

hence the intersection is

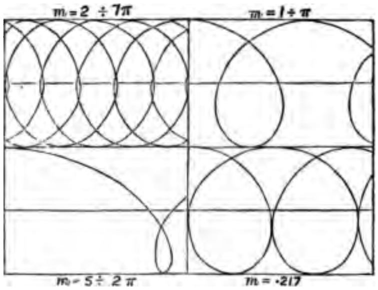
$$x+y+z = \begin{vmatrix} \eta^2+2l\zeta\xi, & \zeta^2+2l\xi\eta \\ \eta^2+2m\zeta\xi, & \zeta^2+2m\xi\eta \end{vmatrix} : \ldots : = \xi(\eta^3-\zeta^3)+\eta(\zeta^3-\xi^3)+\zeta(\xi^3-\eta^3),$$

on reduction.

But this point lies on U; it is, in fact, the tangential of the point (ξ, η, ζ). [See CAYLEY's *Memoir on Cubics*, or SALMON's *Higher Plane Curves*.]

7631. (By the late Professor CLIFFORD, F.R.S.)—A point moves uniformly round a circle while the centre of the circle moves uniformly with less velocity along a straight line in its plane; find the nodes of the curve which the point describes.

Solution by G. HEPPEL, M.A.; G. B. MATHEWS, B.A.; *and others.*



Let the path of the centre be the axis of x, and let the point be always supposed to start from the radius perpendicular to this, which is the axis of y. Then, if m be the ratio of the velocities, and the radius of the circle is taken as unity, the curve is given by $y = \cos\theta$, $x = m\theta + \sin\theta$. Hence the form of the curve depends solely on m. In the limit, when $m = 1$ the curve consists of a series of cycloids. The condition of a node is that $m\theta + \sin\theta = m(2k\pi-\theta) + \sin(2k\pi-\theta)$, or $\sin\theta = m(k\pi-\theta)$.

Now first suppose a line of nodes on the axis of x, then

$$\theta = \tfrac{1}{2}\pi,\; m(k\pi - \tfrac{1}{2}\pi) = 1, \text{ therefore } m = \frac{2}{(2k-1)\pi}.$$

Again, $\dfrac{dy}{dx} = \dfrac{-\sin\theta}{m+\cos\theta}$, and this becomes infinite if $\cos\theta = -m$. Hence, if loops touch at all, they touch below the axis of x, and the conditions of touching are that $\sin\theta = m(k\pi-\theta)$; $\cos\theta = -m$. Solving this approximately, if $k = 1$, $m = \cdot 217$; if $k = 2$, $m = \cdot 129$.

We thus arrive at the following results: $m = 1$, a series of cycloids; between 1 and $\frac{2}{\pi}$, one line of nodes below axis of x; $m = \frac{2}{\pi}$, one line on axis; between $\frac{2}{\pi}$ and $\cdot 217$, one above; $m = \cdot 217$, one above and loops touching; between $\cdot 217$ and $\frac{2}{3\pi}$, two below, one above; $m = \frac{2}{3\pi}$, one below, one on axis, and one above; between $\frac{2}{3\pi}$ and $\cdot 129$, one below, two above; $m = \cdot 129$, one below, two above, and loops touching; between $\cdot 129$ and $\frac{2}{5\pi}$, three below, two above, and so on.

The figure gives four examples for different values of m.

[Mr. HEPPEL thinks that some of the curves obtained in the way suggested in the Question might be utilized for Art-purposes.]

7597. (By Professor TOWNSEND, F.R.S.)—A system of plane waves, propagated by small parallel and equal rectilinear vibrations, being supposed to traverse in any direction an isotropic elastic solid, under the action of its internal elasticity only; show that the direction of vibration is necessarily either parallel or perpendicular to that of propagation, and determine the velocities of the latter corresponding to the two cases.

Solution by the PROPOSER.

Denoting by ξ, η, ζ the small displacements at any point x, y, z of the solid, by μ and ν its coefficients of resistance to compression and distortion respectively, and by ρ its density supposed approximately constant throughout the motion; the equations of motion of any small disturbance propagated through it by virtue of its internal elasticity only are, as is well known,

$$\rho\frac{d^2\xi}{dt^2} = (\mu+\tfrac{1}{3}\nu)\frac{d\omega}{dx} + \nu\nabla_2\xi, \quad \rho\frac{d^2\eta}{dt^2} = (\mu+\tfrac{1}{3}\nu)\frac{d\omega}{dy} + \nu\nabla_2\eta,$$

$$\rho\frac{d^2\zeta}{dt^2} = (\mu+\tfrac{1}{3}\nu)\frac{d\omega}{dz} + \nu\nabla_2\zeta,$$

where ω is the cubical dilatation $\frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}$, and ∇_2 the familiar symbol of operation $\left(\frac{d}{dx}\right)^2+\left(\frac{d}{dy}\right)^2+\left(\frac{d}{dz}\right)^2$, at the point x, y, z of the solid.

Supposing now these equations to be satisfied for a system of plane waves propagated as in the question, and represented in consequence by the equations, $\xi = k\cos\alpha\,.f(lx+my+nz-vt) = k\cos\alpha\,.f(\phi)$,

$$\eta = k\cos\beta\,.f(\phi), \quad \zeta = k\cos\gamma\,.f(\phi),$$

where α, β, γ are the direction angles and k a small constant representing the absolute magnitude of vibration, l, m, n, and v the direction cosines and the velocity of propagation, and f any arbitrary periodic function oscillating between extreme limits of finite magnitude, and representing in consequence vibratory motion; we get, by substitution in the equations of propagation, the three following equations of connection between the several magnitudes involved, viz.,

$$\rho v^2\cos\alpha = (\mu+\tfrac{1}{3}\nu)\, l\,(l\cos\alpha+m\cos\beta+n\cos\gamma)+\nu\cos\alpha,$$

$$\rho v^2\cos\beta = (\mu+\tfrac{1}{3}\nu)\, m\,(l\cos\alpha+m\cos\beta+n\cos\gamma)+\nu\cos\beta,$$

$$\rho v^2\cos\gamma = (\mu+\tfrac{1}{3}\nu)\, n\,(l\cos\alpha+m\cos\beta+n\cos\gamma)+\nu\cos\gamma,$$

which give α, β, γ and v, when determinable, in terms of l, m, n, μ, ν, and ρ, which are supposed to be all given or known.

By elimination of v^2 between these latter equations in pairs, we get immediately the three following equations of connection between α, β, γ and l, m, n, viz.,

$$(m\cos\gamma-n\cos\beta)\,(l\cos\alpha+m\cos\beta+n\cos\gamma) = 0,$$

$$(n\cos\alpha-l\cos\gamma)\,(l\cos\alpha+m\cos\beta+n\cos\gamma) = 0,$$

$$(l\cos\beta-m\cos\alpha)\,(l\cos\alpha+m\cos\beta+n\cos\gamma) = 0,$$

which show at once that, either

$$\cos\alpha : \cos\beta : \cos\gamma = l : m : n, \quad \text{or} \quad l\cos\alpha+m\cos\beta+n\cos\gamma = 0,$$

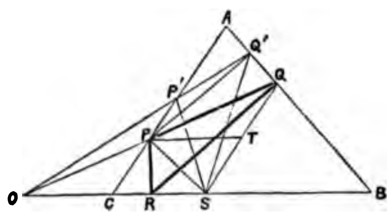
and establish in consequence the first part of the question.

Solving for v^2 in the two cases respectively, and denoting the corresponding velocities by v_n and v_t as corresponding to normal and to transversal vibrations respectively, we get, in answer to the second part of the question, that $(v_n)^2 = \frac{\mu + \frac{4}{3}\nu}{\rho}$ and that $(v_t)^2 = \frac{\nu}{\rho}$; which show that the former depends on the two coefficients μ and ν, and is always greater than the latter, which depends only on the coefficient ν of the substance.

That v_t should depend only on ν, and that v_n should on the contrary depend on both μ and ν, would appear also *à priori* from the obvious consideration that, while transversal vibrations can from their nature produce only change of form, normal vibrations, when other than those of the entire mass as a rigid whole in its space, must on the contrary produce at once changes of volume and of form in the molecules of the substance.

7638. (By the Editor.)—If from a given point O, in the prolongation through C of the base BC of a given triangle ABC, a straight line OPQ be drawn, cutting the sides AC, AB in P, Q; show that, R being any point in the base, the triangle PQR will be a maximum when a parallel QS to AC through Q cuts BC in a point S, such that OS is a mean proportional between OB and OC.

Solutions by (1) A. H. Curtis, LL.D., D.Sc.; (2) G. Heppel, M.A.



1. Let S be the point such that $OS^2 = OB \cdot OC$; then, if SQ be drawn parallel to CA, SP will be parallel to BA, since

$$OQ : OP = OS : OC = OB : OS;$$

and, if OP′Q′ be any other cutting line, we have

$$\Delta SQP = SQ'P > SQ'P';$$

hence the triangle PSQ is the maximum of all triangles that have a common vertex at S, their base angles on AC, AB, and their bases passing through O; moreover, $\Delta P'RQ' : P'SQ' =$ distance of R from P′Q′ : distance of S from P′Q′ = OR : OS = a known ratio, hence these two triangles are together maxima.

2. Draw PT parallel to CB, and let it meet QS, the parallel through Q to AC, in T; then, putting $OB = b$, $OC = c$, $AC = h$, $OS = x$, the maximum value of ΔPQR depends upon that of $QS \cdot OR - PC \cdot OR$, or on that of $QS - PC$, or QT. Now we have

$$QS = \frac{h(b-x)}{b-c}, \text{ and } QT = \frac{h(b-x)(x-c)}{(b-c)x};$$

hence the maximum value depends upon that of $b+c-\frac{bc}{x}-x$, or of

$$(b^{\frac{1}{2}}-c^{\frac{1}{2}})^2-(b^{\frac{1}{2}}c^{\frac{1}{2}}x^{-\frac{1}{2}}-x^{\frac{1}{2}})^2;$$

and this evidently occurs when $x = b^{\frac{1}{2}}c^{\frac{1}{2}}$, or $x^2 = bc$, that is to say, when OS is a mean proportional between OB and OC.

7644. (By W. S. McCAY, M.A.)—Prove that the three lines that join the mid-point of each side of a triangle to the mid-point of the corresponding perpendicular meet in a point.

Solutions by (1) A. H. CURTIS, LL.D., D.Sc.; (2) HAROLD HARLEY, B.A.

1. The trilinear coordinates of the middle point of the side c, the axes being the sides of the triangle, are $\frac{1}{2}c\sin B$, $\frac{1}{2}c\sin A$, 0, while those of the middle point of the corresponding perpendicular p_3 are $\frac{1}{2}p_3\cos B$, $\frac{1}{2}p_3\cos A$, $\frac{1}{2}p_3$, and the equation of the line joining these points is

$$x\sin A-y\sin B-z\sin(A-B) = 0 \quad\text{.........................(1)},$$

while those of the two corresponding lines are

$$y\sin B-z\sin C-x\sin(B-C)=0,\ z\sin C-x\sin A-y\sin(C-A)=0\text{...(2, 3)}.$$

If we multiply (1) by sin 2C, (2) by sin 2A, (3) by sin 2B, and add, the coefficients of x, y, z vanish identically; hence the lines meet in a point.

Again, if we add (2) and (3), we obtain $x\sin B-y\sin A = 0$..........(4) as the equation of a line passing through the intersection of (2) and (3), and obviously through the vertex C, while

$$y\sin C-z\sin B = 0 \text{ and } z\sin A-x\sin C = 0 \text{ .... (5, 6)}$$

are the equations of the two corresponding lines through the common point of (1), (2), (3), and the vertices A and B. The form of the equation (4) shows that the line which it represents makes with the side a the same angle which the bisector of the side c makes with b; and hence it follows that the point of intersection of (1), (2), (3) is the second focus of the ellipse [BROCARD's ellipse] inscribed in the triangle, and having for its first focus the point of intersection of the three lines joining the three vertices to the middle points of the opposite sides.

2. Let D, E, F be the mid-points of the sides of the triangle ABC, and G, H, K the mid-points of the perpendiculars from A, B, C on the sides of the triangle DEF; then, since ED is parallel to AB, therefore the right-angled triangles CEK, BFH are similar; hence we have

$$\frac{EK}{FH}=\frac{EC}{BF}=\frac{b}{c},\quad \frac{FG}{DK}=\frac{c}{a},\quad \frac{DH}{EG}=\frac{a}{b};$$

therefore DH . FG . EK = DK . EG . FH; whence, by CEVA's theorem, DG, EH, FK meet in a point.

[Mr. TUCKER remarks that the theorem in the Question is given in §11, p. 7, of NEUBERG's paper "Sur le Centre des Médianes antiparallèles," the point of intersection being what is known as the Point de Grèbe, or Symmedian point, of the triangle,—a fact unknown to the PROPOSER, who obtained the theorem from the three rectangles in Question 7612 having a common centre (the Symmedian point). The property in the question may be more generally enunciated as follows:—"If the mid-points of the portions intercepted on any three concurrent lines from the vertices of a triangle, between these and the opposite sides, be joined to the mid-points of the corresponding sides, the three connectors will pass through the same point," —a theorem which may be proved thus:—If D, E, F be the mid-points of the sides and AG, BH, CK any three concurrent lines meeting the sides in G, H, K, and g, h, k be the mid-points of AG, BH, CK; then g, h, k lie on EF, FD, DE respectively, and $Fg = \frac{1}{2}BG$, $gE = \frac{1}{2}GC$, $Ek = \frac{1}{2}Ak$, $kD = \frac{1}{2}KB$, &c. But $BG.CH.AK = GC.HA.KB$; hence $Fg.Dh.Ek = gE.hF.kD$, therefore &c.]

4516. (By the late T. COTTERILL, M.A.)—In a spherical triangle, of the five products

$$\cos a \cos A, \quad \cos b \cos B, \quad \cos c \cos C, \quad \cos a \cos b \cos c, \quad -\cos A \cos B \cos C,$$

one is negative, the other four being positive. In the solution of such triangles, what parts must be given that the affections of the remaining three can be determined by this theorem?

Solution by J. J. WALKER, M.A., F.R.S.

(1) If $\cos a$, $\cos b$, $\cos c$ are all positive, and $a > b > c$; then $\cos A$ alone may be negative, since both $\cos b$, $\cos c > \cos a$ and therefore *à fortiori* $> \cos c \cos a$, $\cos b \cos a$. But $-\cos A \cos B \cos C$ is opposite in sign to $\cos A$. Hence either the first or last of the five products alone will be negative.

(2) If $\cos a$ alone is negative, then $\cos B \cos C$ are both positive, but $\cos A$ is negative. Hence of the five products $\cos a \cos b \cos c$ alone will be negative.

(3) If $\cos a$, $\cos b$ are both negative, but $\cos c$ is positive, $a > b$; then $\cos A$ must be, and $\cos B$ may be, negative, $\cos C$ must be positive. Hence, of the five products, either $\cos b \cos B$ or $-\cos A \cos B \cos C$ alone will be negative.

(4) If $\cos a$, $\cos b$, $\cos c$ are all negative, then $\cos A$, $\cos B$, $\cos C$ are all necessarily negative. In this case, of the five products, $\cos a \cos b \cos c$ alone can be negative.

It follows from this theorem that, a, b, c being the given parts, if all are $< \frac{1}{2}\pi$, then one only of the three angles can be $> \frac{1}{2}\pi$; but if a, alone, $> \frac{1}{2}\pi$, then a must be $>$, B, C $< \frac{1}{2}\pi$; if two only of the given parts, as a, b, $> \frac{1}{2}\pi$, then one of the two angles A, B must, both may be, $> \frac{1}{2}\pi$; if all three of the given parts are $> \frac{1}{2}\pi$, then all three of the angles A, B, C must also be $> \frac{1}{2}\pi$.

The same things may be predicated *vice versâ* of angles and sides, save that, if two only of the given angles are obtuse, the opposite sides must also be obtuse, otherwise three of the five products would be negative.

In the other cases of solution, in which the theorem gives some clue to the affections of the parts to be found, the reservations are too numerous to make its application useful. . . . " His saltem accumulem donis. . . . "

7550. (By J. GRIFFITHS, M.A.)—If $t = \frac{7}{2} + \frac{3}{4}\,\mathrm{sn}\,u\,.\,\mathrm{sn}\,(K-u)$ and modulus $= \frac{1}{2}\sqrt{3}$, $K = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1-\frac{3}{4}\sin^2\theta)^{\frac{1}{2}}}$; show that

$$\frac{dt}{[(t-2)(t-3)(t-4)(t-5)]^{\frac{1}{2}}} = du.$$

Solution by G. B. MATHEWS, B.A.; D. EDWARDES; *and others.*

$$t-\tfrac{7}{2} = \tfrac{3}{4}\,\mathrm{sn}\,u\,\mathrm{sn}\,(K-u) = \tfrac{3}{4}\,\frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u},\quad t^2-7t+\tfrac{49}{4} = \tfrac{9}{16}\,\frac{\mathrm{sn}^2 u\,(1-\mathrm{sn}^2 u)}{1-\frac{3}{4}\,\mathrm{sn}^2 u},$$

$$(t-2)(t-5) = t^2-7t+10 = \tfrac{9}{16}\,\frac{s^2(1-s^2)}{1-\frac{3}{4}s^2} - \tfrac{9}{4} = -\tfrac{9}{16}\,\frac{(2-s^2)^2}{d^2},$$

$$(t-3)(t-4) = t^2-7t+12 = \tfrac{9}{16}\,\frac{s^2(1-s^2)}{d^2} - \tfrac{1}{4} = -\tfrac{1}{16}\,\frac{(2-3s^2)^2}{d^2},$$

therefore $\quad [(t-2)(t-3)(t-4)(t-5)]^{\frac{1}{2}} = \frac{3}{16}\,\frac{(2-s^2)(2-3s^2)}{d^2};$

but $\quad \dfrac{dt}{du} = \tfrac{3}{4}\,\dfrac{(\mathrm{cn}^2 u - \mathrm{sn}^2 u)\,\mathrm{dn}^2 u + k^2\,\mathrm{sn}^2 u\,\mathrm{cn}^2 u}{\mathrm{dn}^2 u}$

$$= \tfrac{3}{4}\,\frac{(1-2s^2)(1-\frac{3}{4}s^2) + \frac{3}{4}s^2(1-s^2)}{1-\frac{3}{4}s^2}$$

$$= \tfrac{3}{16}\,\frac{4-8s^2+3s^4}{1-\frac{3}{4}s^2} = \tfrac{3}{16}\,\frac{(2-s^2)(2-3s^2)}{d^2} = [(t-2)(t-3)(t-4)(t-5)]^{\frac{1}{2}}.$$

Otherwise :—Referring to Mr. GRIFFITHS's paper in the *Proc. Math. Soc.*, Feb. 8th, 1883, putting therein $\alpha=5$, $\beta=4$, $\gamma=2$, $\delta=3$, then $k=\frac{1}{2}\sqrt{3}$, $k'=\frac{1}{2}$, and $\mathrm{cn}\,u_0=0$, therefore $u_0 = K$ mod. $\frac{1}{2}\sqrt{3}$; also $M=2$ and

$$t = \tfrac{7}{2} - \tfrac{3}{2}\,\mathrm{cn}\,u\,\mathrm{cn}\,(K-u),$$

or, since $\mathrm{cn}\,u\,\mathrm{cn}\,(K-u) = k'\,\mathrm{sn}\,u\,\mathrm{sn}\,(K-u)$, $t = \frac{7}{2} - \frac{3}{4}\,\mathrm{sn}\,u\,\mathrm{sn}\,(K-u)$,

and $\quad dt \div (t-2\,.\,t-3\,.\,t-4\,.\,t-5)^{\frac{1}{2}} = -du.$

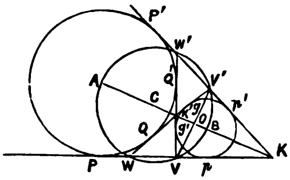
[Or, again, by putting $\alpha=5$, $\beta=4$, $\gamma=3$, $\delta=2$, the formula

$$\frac{dy}{[(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)]^{\frac{1}{2}}} = \frac{2}{[(\alpha-\gamma)(\beta-\delta)]^{\frac{1}{2}}}\cdot\frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}},$$

$$k^2 = \frac{\beta-\gamma\,.\,\alpha-\delta}{\alpha-\gamma\,.\,\beta-\delta}.\Big]$$

7511. (By Professor WOLSTENHOLME, M.A., Sc.D.)—A, B are the given centres of two circles; Pp, P$'p'$ the external common tangents, Qq, Q$'q'$ the internal common tangents, P, Q being on the same side of the axis; Pp, Q$'q'$ intersect at right angles in V, and P$'p'$, Qq at right angles in V$'$: prove that (1) P, Q, q', p' lie on one straight line, P$'$, Q$'$, q, p on another straight line, whose directions are fixed, and these two straight lines and VV$'$ meet in one point O; (2) the common tangents Pp, P$'p'$ are equal to the sum of the radii, and Qq, Q$'q'$ to the difference; (3) the points of contact lie on four fixed circles, and the common tangents pass through two fixed points; (4) PQ$'$, P$'$Q, pq', $p'q$ all intersect in one fixed point C bisecting AB; (5) PQ, P$'$Q$'$, pq, $p'q'$ are all of equal length, and the ratio Pp' : Qq' is the duplicate ratio of Pp : Qq; (6) the ratios OP : p'O, OQ : Oq' are equal, and are equal to the ratio of the radii of the two circles; (7) the common tangents and the two straight lines through the eight points of contact all touch the same parabola, focus C, and directrix VV$'$.

Solution by W. J. C. SHARP, M.A.



Let AB meet Pp and Qq in K and K$'$ respectively, and let Qq meet Pp in W; then the points P$'$, p', Q$'$, q', V$'$, W$'$ are the reflexions of P, p, Q, q, V, W with respect to AB, — for the one-half of each circle is the reflexion of the other with respect to the same line; and, if C be the middle point of AB, the circle with centre C, passing through A and B, will also pass through V and W and their reflexions V$'$ and W$'$, for VB bisects the angle q'Vp, and VA bisects q'VA, and AVB is a right angle; similarly AWB is a right angle. Hence WW$'$ passes through C, and it and PP$'$, pp', QQ$'$, qq', and VV$'$ are all perpendicular to the line of centres.

Now let VK and VW$'$ be the positive directions of the rectangular Cartesian coordinates, and a, b, c be the radii of the circles with centres A, B, C, so that $(2c)^2 = (a-b)^2 + (a+b)^2 = 2(a^2+b^2)$ or $2c^2 = a^2+b^2$, then the equations to the circles are

$$x^2+y^2+2ax-2ay+a^2 = 0 \equiv A, \quad x^2+y^2-2bx-2by+b^2 = 0 \equiv B,$$

and $$x^2+y^2+(a-b)\,x-(a+b)\,y = 0 \equiv C,$$

and therefore W is the point $-(a-b)$, 0, and W$'$ is 0, $a+b$, and the polar of W with respect to A is $bx-ay+ab = 0$, which is also that of W$'$ with respect to B, and PQ$q'p'$ is a straight line, and therefore its reflexion P$'$Q$'qp$, the polars of W with respect to B and of W$'$ with respect to A, $ax+by-ab = 0$ and these meet on $(a-b)\,y = (a+b)\,x$, the perpendicular VOV$'$ from the origin on $y-b = -\dfrac{a-b}{a+b}(x-b)$, the line of centres and at the point O where these meet. This proves (1), and (2) follows at once from the values of the coordinates of the points of contact.

Evidently the circle through any two of the points P, Q, q', p', and with its centre in AB, passes through the reflexions of the two points, and there are six such circles, the two given circles being two of them (3). VB cuts the polar of V with respect to B (pq') at right angles, and, as they are the diagonals of the square $V\rho Bq'$, bisects it; then pq' passes through C, and similarly so do PQ', P'Q, and $p'q$ (4). Again, $AW = BW = c\sqrt{2}$,

therefore $PQ = 2\dfrac{AP \cdot PW}{WA} = 2\dfrac{ab}{c\sqrt{2}}$ and $pq = 2\dfrac{Bp \cdot Wp}{WB} = 2\dfrac{ab}{c\sqrt{2}}$,

therefore $PQ = pq$, and these are equal to their reflexions $P'Q'$, $p'q'$.

And $\quad OP : p'O = PK : p'K = a : b$,

$\quad OQ : q'O = QK' : K'q = a : b$, which proves (6),

and $\quad Pp' : PO = a+b : a$ and $Qq' : QO = a-b : a$,

therefore $\quad Pp' : Qq' = (a+b)\, PO : (a-b)\, QO$,

and $PO : QO = Pp : Qq$ because the triangles PpO, QqO are similar, therefore $Pp' : Qq' :: Pp^2 : Qq^2$, which completes the proof of (5) and shows that the lines $PQq'p'$ and $pqQ'P'$ cut at right angles, and therefore touch the parabola in (7), as do the common tangents for the same reason.

The property, that the circle on the line of centres passes through the four points of intersection of the common tangents to two circles, which are not centres of similitude, is true at whatever angle the common tangents intersect.

The middle points of W'V', W'V, WV', WV all lie on the radical axis of the two circles, which is the tangent at the vertex of the parabola.

C is the centre of the circles $PP'p'p$ and $QQ'qq'$, two of the four circles, since WV and Pp and W'V' and $P'p'$ are bisected in the same points as are VW' and $Q'q'$ and V'W and Qq.

Again, $\quad Cq : Cp' = \frac{1}{2} W'V' : \frac{1}{2} W'V' + V'p' = a-b : a+b :: Qq : Pp$,

and $\quad Cq : Cp' = Cq' : Cp = CQ : CP' = CQ' : CP$,

also VOV' is the polar of C with respect to B;

therefore $\quad BO \cdot BC = b^2$ and therefore $BO = \dfrac{b^2}{c}$,

and $\quad AO \cdot AC = (AB - AO)\, BC = 2c^2 - b^2 = a^2$;

and WW' is the polar of O with respect to A, and O and C are inverse points with respect to both circles; therefore, by Question 7209, the four circles described upon the common tangents all pass through these points, and the circles A and B will reciprocate, about either of these points, into confocal conics.

7648. (By D. Biddle.)—A series of isosceles triangles, beginning with the equilateral, is such that each in succession has two-thirds the vertical angle and two-thirds the base of its predecessor. Show that, when the base and vertical angle reach zero, the height of the last in the series is to the height of the first as $2\sqrt{3} : \pi$.

Solution by W. G. Lax, B.A.; Sarah Marks; *and others.*

Let $2a$ be the base of the first triangle, its vertical angle being $\frac{1}{3}\pi$ and height $= a \cot \frac{1}{6}\pi$; then the height of the $(n+1)^{\text{th}}$ triangle of the series $= (\frac{2}{3})^n\, a \cot [(\frac{2}{3})^n \frac{1}{6}\pi]$, and therefore the required ratio is

$$\mathrm{L}^t_{n=\infty} \frac{(\frac{2}{3})^n a \cot [(\frac{2}{3})^n \frac{1}{6}\pi]}{a \cot \frac{1}{6}\pi} = \mathrm{L}^t_{n=\infty} \frac{(\frac{2}{3})^n}{\sin [(\frac{2}{3})^n \frac{1}{6}\pi]} \frac{\cos [(\frac{2}{3})^n \frac{1}{6}\pi]}{\cot \frac{1}{6}\pi}$$

$$= \frac{1}{\sqrt{3}\, \frac{1}{6}\pi} \mathrm{L}^t_{n=\infty} \frac{(\frac{2}{3})^n \frac{1}{6}\pi}{\sin [(\frac{2}{3})^n \frac{1}{6}\pi]} = \frac{6}{\pi\sqrt{3}} = \frac{2\sqrt{3}}{\pi}.$$

[The PROPOSER remarks that, if a series of sectors of any circle be taken, with angles similarly diminishing to zero from 60°, the arcs will bear the same ratio to one another that the bases of the triangles in the question do; so that, if we suppose the height of the last of the series of triangles to correspond with the radius of the circle, $= 1$, the base of the first in the series will be $\frac{1}{3}\pi$ and its height $\frac{1}{6}\pi\sqrt{3}$; thus the ratio is

$$1 : \tfrac{1}{6}\pi\sqrt{3} = 2\sqrt{3} : \pi.]$$

7635. (By Professeur ANGELOT.)—Démontrer que

$$\tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{8} + \tan^{-1}\tfrac{1}{18} + \ldots + \tan^{-1}\frac{1}{2n^2} + \ldots \textit{ad. inf.} = \tfrac{1}{4}\pi.$$

Solution by R. KNOWLES, B.A., L.C.P.; J. O'REGAN; *and others.*

It is easy to prove that the respective sums of 2, 3, 4 ... r terms are

$$\tan^{-1}\tfrac{2}{3},\ \tan^{-1}\tfrac{3}{4},\ \tan^{-1}\tfrac{20}{21}, \ldots \tan^{-1}\frac{r}{r+1};$$

hence the sum to infinity $= \tan^{-1} 1 = \frac{1}{4}\pi$.

7628. (By R. KNOWLES, B.A., L.C.P.)—If a, b, c represent the sides of a triangle, and $s_1 = s - a$, &c., prove that

$$bc - s_1^2 = ac - s_2^2 = ab - s_3^2 = r(r_1 + r_2 + r_3).$$

Solution by CHRISTINE L. FRANKLIN, B.A.; *and* W. J. GREENSTREET, B.A.

From $r = s_2 \tan \frac{1}{2}\mathrm{B}$, $r_1 = s_3 \cot \frac{1}{2}\mathrm{B}$, &c., we have

$$r(r_1 + r_2 + r_3) = s_2 s_3 + s_3 s_1 + s_1 s_2 = s_2 s_3 + a s_1$$
$$= \tfrac{1}{2}(ab + bc + ca) - \tfrac{1}{4}(a^2 + b^2 + c^2) = bc - s_1^2 = \text{\&c.}$$

7623. (By the EDITOR.)—If a knight is placed in a given square on a chess-board, show (1) how to move it 63 times, so that it may not occupy any square twice; and (2) how to solve the same problem when the number of squares is 49 or 81.

I. *Solution by* M. JENKINS, M.A.

In the problem of the knight's move I propose to show how to correct an imperfect arrangement of the moves by a method which I have never found to fail in any example where the side of the square has more than 5 places.

I have been told that the automaton at the Crystal Palace would, offhand and very quickly, move the knight in the required manner, starting from any square which the spectator chose.

This would seem to indicate that the automaton had a simple rule for avoiding a false move.

The nearest approach I can make to such a rule in the case of the ordinary chess-board is "Start from a corner, and keep to the outside, not going into a corner unless the inlet and outlet are both unoccupied." This would give us 48 of the moves in the annexed square (Fig. 1), which shows an unbroken chain of moves. Since the 64 is a knight's move from 1, if we could commit the order of the numbers in Fig. 1 to memory, we could imitate the automaton. There is a defect in the tactical rule as stated, since it would lead us to go from 48 into 53 rather than into 49; but it would guide us fairly well from 49 to 64, if, for the purpose of ascertaining the outside, we suppose the rows and columns which have been completely filled to be cut off.

I will now explain the method I have referred to, which will help us where the imperfect rule fails.

Fig. 2, taken from the *Illustrated London News*, shows a broken chain of moves, which may be divided into 2 endless chains, viz., from 1 to 32, and from 33 to 64.

If we call two squares which are a knight's move from each other a link, there is a link in one of the two chains which may be connected with a link in the other chain, viz., 26, 27 with 59, 60. If therefore we pass from one chain to the other by means of the links, following the figures in the order indicated by the arrows appended to the circles accompanying Fig. 2, we shall obtain a single endless chain (Fig. 3).

For the square whose side has 6 places, if we start from a corner and keep to the outside, subject to the corner rule, we shall be able to move 34 moves without hindrance; the two squares which are left happen to be in connexion with the first corner. Joining them on and moving the figures two places backwards, we obtain a single broken chain, which can be divided into two endless chains, and then converted into a single endless chain (Fig. 4) just as in the previous example.

If the first trial is a bad one consisting of several chains broken or endless, with several detached single squares, I have found no difficulty in reducing these down to a single endless chain in the case of a square whose side has an even number of places >4, or to a single broken chain in the case of a square whose side has an odd number of places >5.

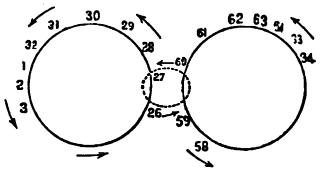
A knight could not move in a single endless chain in a square whose side has an odd number of places, because on a chequered board the colour of the square changes at each move. In a square of 49 places, the 49 would be of the same colour as 1, and could not therefore be a knight's move from it.

For the square of 49 places, starting from a corner as before, I get the square (Fig. 5) filled up with a broken chain 1 to 45, a link 46, 47, and another link 48, 49. The link 46, 47 can be connected with the link 32, 33, and the link 48, 49 with the link 36, 37, whence we obtain the single broken chain (Fig. 6).

For the square of 81 places, I first obtain Fig. 7, containing 3 broken chains, viz., from 1 to 73, from 74 to 77, and from 78 to 81. The extremities 81, 78 of the third chain may be connected with the first chain by means of the link 16, 17, and the second chain with the first chain by means of the link 69, 70, thus obtaining Fig. 8, which is a single unbroken chain.

26	23	38	9	36	21	48	7
39	10	25	22	53	8	35	20
24	27	12	37	60	49	6	47
11	40	63	52	57	54	19	34
28	13	56	59	50	61	46	5
41	64	51	62	55	58	33	18
14	29	2	43	16	31	4	45
1	42	15	30	3	44	17	32

FIG. 1.



With Figs. 2 and 3.

48	57	22	27	50	55	20	29
23	26	49	56	21	28	51	54
58	47	24	17	60	53	30	19
25	16	59	46	31	18	61	52
36	45	2	15	62	43	32	9
3	14	37	44	1	8	63	42
38	35	12	5	40	33	10	7
13	4	39	34	11	6	41	64

FIG. 2.

38	29	22	59	36	31	20	61
23	26	37	30	21	60	35	32
28	39	24	17	58	33	62	19
25	16	27	40	63	18	57	34
50	41	2	15	56	43	64	9
3	14	49	42	1	8	55	44
48	51	12	5	46	53	10	7
13	4	47	52	11	6	45	54

FIG. 3.

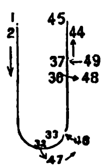
18	15	20	29	36	13
21	30	17	14	7	28
16	19	22	1	12	35
23	2	31	8	27	6
32	9	4	25	34	11
3	24	33	10	5	26

FIG. 4.

9	30	19	38	7	28	17
20	39	8	29	18	37	6
31	10	43	49	47	16	27
40	21	32	45	42	5	36
11	44	41	46	48	26	15
22	33	2	13	24	35	4
1	12	23	34	3	14	25

FIG. 5.

With Figs. 5 and 6.



The loop marks the starting-point of a chain.

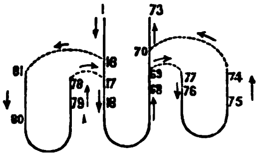
9	30	19	42	7	28	17
20	43	8	29	18	41	6
31	10	47	40	33	16	27
44	21	32	49	46	5	38
11	48	45	34	39	26	15
22	35	2	13	24	37	4
1	12	23	36	3	14	25

FIG. 6.

13	28	41	54	11	26	39	52	9
42	55	12	27	40	53	10	25	38
29	14	57	68	81	75	63	8	51
56	43	16	**74**	64	73	80	37	24
15	30	67	58	69	62	76	50	7
44	17	70	65	60	79	72	23	36
31	66	59	**78**	71	77	61	6	49
18	45	2	33	20	47	4	35	22
1	32	19	46	3	34	21	48	5

Fig. 7.

With Fig. 7.



The loop marks the starting-point of a chain.

13	32	45	58	11	30	43	56	9
46	59	12	31	44	57	10	29	42
33	14	61	72	17	76	67	8	55
60	47	16	77	68	81	18	41	28
15	34	71	62	73	66	75	54	7
48	21	78	69	64	19	80	27	40
35	70	63	20	79	74	65	6	53
22	49	2	37	24	51	4	39	26
1	36	23	50	3	38	25	52	5

Fig. 8.

II. *Solution by* D. Biddle.

(1) If A represent the *column* in which the knight stands, and B the *row*, then $A \pm 1$, $B \pm 2$, or $A \pm 2$, $B \pm 1$ will represent the position at the next move. The annexed diagram (Fig. 9) shows how on an ordinary chess-board the knight may proceed to every square once, and return to the square from which he started, which may be in any position, the cycle being complete. If we denote the columns by the first figures and the rows by the second figures in a series of numbers, we obtain the following:—

16	45	30	5	18	43	32	7
29	4	17	44	31	6	19	42
46	15	62	59	52	55	8	33
3	28	53	56	61	58	41	20
14	47	60	63	54	51	34	9
27	2	25	12	57	38	21	40
48	13	64	37	50	23	10	35
1	26	49	24	11	36	39	22

Fig. 9.

11, 23, 15, 27, 48, 67, 88, 76, 84, 72, 51, 43, 22, 14, 26, 18, 37, 58, 77, 85, 73, 81, 62, 41, 33, 21, 13, 25, 17, 38, 57, 78, 86, 74, 82, 61, 42, 63, 71, 83, 75, 87, 68, 47, 28, 16, 24, 12, 31, 52, 64, 56, 35, 54, 66, 45, 53, 65, 46, 34, 55, 36, 44, 32,

in which each figure up to 8 occurs 8 times as first and 8 times as second, and in which the successive numbers are formed by the addition of $\pm(10 \pm 2)$ or $\pm(20 \pm 1)$. Now $(10+2)$ is added 9 times and deducted 9 times; $(10-2)$ is added 9 times also and deducted 9 times; $(20+1)$ is added 8 times and deducted 8 times (if the knight completes the cycle by returning to its original square); and (20−1) is added 6 times and deducted 6 times. The figures denoting change of column amount to 46−46, those denoting change of row to 50−50. The balance is perfect. But, since the arrangement is unsymmetrical, it is evident that a difference in the course can be effected by simply starting from each of the 4 corners in rotation, by taking either of the two directions which lead out of the corners, and also in each of these 8 cases by reversing the course. Thus from one primary arrangement we obtain 16 distinct routes by which the knight can complete the round of the board and return to the square from which he started. We need not here consider the number of primary arrangements that could be made of this kind. But we may point out that to fulfil the requirements of the problem, as regards the ordinary chess-board, it is not necessary to be able to return to the original square. In Fig. 1 it is easy to see that by going forwards from 1 to 27 and then backwards from 64 to 28, we could finish on a remote square and yet traverse the whole board as required. Similarly, we could finish on 12, 48, or 58; and it is not improbable that, by modification of some one of the several primary arrangements (each with its 16 distinct routes), we could begin and end on any two specified squares of different colours.

(2) Where the number of squares on the board is odd, as in the given instances, 49 or 81, a complete cycle seems impracticable; that is, the knight cannot return to the square from which he started. The balance between the outgoing and return moves is necessarily imperfect, where there cannot be an equal number of each. But it is quite possible to comply with the requirements of the problem in regard to a 7^2 board, so far at least as 25 out of the 49 starting-points are concerned. The following diagrams (Figs. 10, 11) give two arrangements from which the 25 tours mentioned can easily be mapped out:—

27	16	5	46	25	14	3
6	47	26	15	4	45	24
17	28	35	40	37	2	13
48	**7**	38	**1**	34	23	44
29	18	41	36	39	12	33
8	**49**	20	31	10	43	22
19	30	9	42	21	32	11

FIG. 10.

19	4	29	6	21	8	11
28	37	20	39	10	31	22
3	18	5	30	7	12	9
36	**27**	38	17	40	23	32
45	2	47	26	**43**	16	13
48	35	44	41	14	33	24
1	46	**49**	34	25	42	15

FIG. 11.

Routes.	No. of similar starting-points.
1—49 (Fig. 10)	1
1—49 (Fig. 11)	4
49—1 (Fig. 10)	4
49—1 (Fig. 11)	8
27—49, 26—1 (Fig. 11)	4
43—1, 44—49 (Fig. 11)	4
	25

Treating the squares in the two arrangements as we treated Fig. 1, we find that, in Fig. 10, (10+2) is added 6 times and deducted 6 times, also (20+1) is added 6 times and deducted 6 times; but (10−2) is added 7 times and deducted only 5 times, and $(20-1)$ is added 5 times and deducted 7 times: balance $= -22 = 22-44$ (the terminal squares in the chain). In Fig. 3 $(10+2)$ is added 7 times and deducted 4 times; $(10-2)$ is added 10 times and deducted 7 times; $(20+1)$ is added 4 times and deducted 5 times; $(20-1)$ is added 5 times and deducted 6 times: balance $= +20 = 31-11$. About this latter there seems no regularity, which leads one to imagine that there is no real reason why the knight should be unable to complete his course from the remaining 24 squares of the 7^2 board. But these tours are certainly attended with greater difficulty.

On turning to the 9^2 board, we find that we can divide it into two portions, a central set of 5^2 squares, and an outer fringe two squares deep, and that these two portions can each be entirely traversed by the knight, without crossing the boundary line between the two, provided he start from a corner square. In Fig. 12, the simplest arrangement is laid down, and this would serve for a great number of distinct routes, with but slight modification. In Fig. 13 the two portions are used conjointly.

13	26	39	52	11	24	37	50	9
40	53	12	25	38	51	10	23	36
27	14	59	76	71	66	61	8	49
54	41	70	65	60	77	72	35	22
15	28	75	58	81	62	67	48	7
42	55	80	69	64	73	78	21	34
29	16	57	74	79	68	63	6	47
56	43	2	31	18	45	4	33	20
1	30	17	44	3	32	19	46	5

Fig. 12.

71	54	27	40	69	52	25	38	67
28	41	70	53	26	39	68	51	24
55	72	63	4	9	14	61	66	37
42	29	10	15	62	65	8	23	50
73	56	5	64	3	60	13	36	81
30	43	16	11	58	7	2	49	22
17	74	57	6	1	12	59	80	35
44	31	76	19	46	33	78	21	48
75	18	45	32	77	20	47	34	79

Fig. 13.

7040. (By Rev. T. R. Terry, F.R.A.S.)—If p and q be two positive integers such that $p>q$, and if r be any positive integer, or any negative integer numerically greater than p, show that

$$1-\frac{q}{p-q+1}\cdot\frac{r}{p+r-1}+\frac{q(q-1)}{(p-q+1)(p-q+2)}\cdot\frac{r(r-1)}{(p+r-1)(p+r-2)}-\&c.,$$
$$=\frac{p-q}{p}\cdot\frac{p+r}{p-q+r}.$$

[This identity has been suggested by Professor SYLVESTER's Quest. 6978, but a proof may be given independent of the theorem in that Question.]

Solution by W. J. C. SHARP, M.A.

It is easy to see that the equation holds for all values of q if $r=0$ or 1, and for all values of r if $q=0$ or 1. Suppose it to hold when $q-1$ and $r-1$ are written for q and r.

$$\therefore 1-\frac{q-1}{p-q+2}\cdot\frac{r-1}{p+r-2}+\frac{(q-1)(q-2)}{(p-q+2)(p-q+3)}\cdot\frac{(r-1)(r-2)}{(p+r-2)(p+r-3)}-\&c.$$
$$=\frac{p-q+1}{p}\cdot\frac{p+r-1}{p-q+r},$$

$$\therefore 1-\frac{q}{p-q+1}\cdot\frac{r}{p+r-1}+\frac{q(q-1)}{(p-q+1)(p-q+2)}\cdot\frac{r(r-1)}{(p+r-1)(p+r-2)}-\&c.$$
$$=1-\frac{q}{p-q+1}\cdot\frac{r}{p+r-1}\times\frac{p-q+1}{p}\cdot\frac{p+r-1}{p-q+r}=\frac{p-q}{p}\cdot\frac{p+r}{p-q+r};$$

and therefore by induction it holds for all positive values of q and r so long as $q<p$.

Again, if r be negative, $=-s$ say, the formula becomes

$$1-\frac{q}{p-q+1}\cdot\frac{s}{s-p+1}+\frac{q(q-1)}{(p-q+1)(p-q+2)}\cdot\frac{s(s+1)}{(s-p+1)(s-p+2)}-\&c.$$
$$=\frac{p-q}{p}\cdot\frac{s-p}{s+q-p},$$

which holds for all values of $s>p$ if $q=0$ or $q=1$. Suppose it to hold for $q-1$ and $s+1$, put for q and s; therefore

$$1-\frac{q-1}{p-q+2}\;\frac{s+1}{s-p+2}+\frac{(q-1)(q-2)}{(p-q+1)(p-q+2)}\cdot\frac{(s+1)(s+2)}{(s-p+2)(s-p+3)}-\&c.$$
$$=\frac{p-q+1}{p}\cdot\frac{s-p+1}{s+q-p};$$

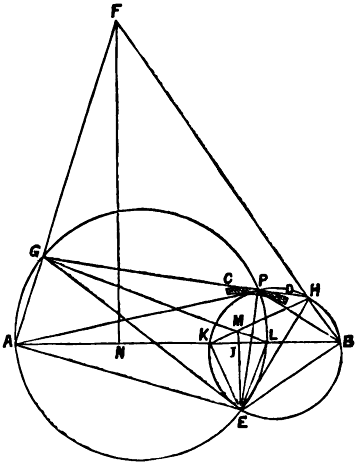
therefore

$$1-\frac{q}{(p-q+1)}\cdot\frac{s}{s-p+1}+\frac{q(q-1)}{(p-q+1)(p-q+2)}\;\frac{s(s+1)}{(s-p+1)(s-p+2)}-\&c.$$
$$=1-\frac{q}{p-q+1}\cdot\frac{s}{s-p+1}\times\frac{p-q+1}{p}\cdot\frac{s-p+1}{s+q-p}=\frac{p-q}{p}\cdot\frac{s-p}{s+q-p},$$

and the formula holds for all values of $s>p$; therefore, &c.

6878, 7422, 7653. (By B. H. RAU, M.A.)—Given a concave spherical mirror, a luminous point, and the position of an eye perceiving one of the reflected rays; find the point of incidence and reflection on the mirror.

Solution by D. BIDDLE; BELLE EASTON; *and others.*



Let A be the luminous point, B the point through which the reflected ray is to pass, CD the concave spherical mirror, and E the centre of its curve. Then $\angle$APE = BPE, and AE, BE are similar chords of the circles EPA, EPB of which EP is a common chord. Draw AF, BF at right angles to AE, BE, and GH tangential to the reflecting surface of the mirror at P. Then, since EPG, EAG in the one circle, and EPH, EBH in the other circle, are right angles, the centres of the circles must be at the mid-points of EG, EH. Moreover, since AE, EB are similar chords, the diameters of the circles must bear the same ratio,

EG : EH = AE : BE,

and the complementary chords also must bear the same ratio, AG : BH = AE : BE.

We also have, given, $\angle$AEB; and AB, with the perpendiculars drawn to it, EI, FN; also AN and BN. Moreover, $\angle$ALE = APE, and $\angle$BKE = BPE. Consequently, EKL is an isosceles triangle and EI bisects KL. Again, ELG and EKH are right angles; and $\angle$EGL = EAB, and $\angle$EHK = EBA. Wherefore, the triangles ELG, EKH are similar to EIA and EIB, the sides being severally as EL : EI; and EM, the prolongation of EI, cuts off equal portions of each, and joins the apices of two triangles EGH, MGH, which have the same base GH, the height of one, EGH, being the radius EP of the curve of the reflecting surface. Now EI . EM = EL². Consequently, if EM be found, and a circle be drawn on EM as diameter, L and K will be its points of intersection with AB. Then G and H can be readily found, and GH will touch the reflecting surface in P, the required point.

Let EM = x, then, since $\angle$IEL = AEG, therefore we have

EA : EG = EI : EL = EL : x and $EA^2 : EA^2 + AG^2 = EI x : x^2 = EI : x$,

hence $$= \frac{EI\,(EA^2 + AG^2)}{EA^2};$$

thus, putting AG = y, EA = 1, EB = a, then, by drawing perpendiculars to AB from G and H, we find

$$GH^2 = \left\{ AB - \left(\frac{AN}{AF} + \frac{BNa}{BF}\right) y \right\}^2 + \left(\frac{FN}{AF} - \frac{FNa}{BF}\right)^2 y^2.$$

Let $$f = \frac{AN}{AF} + \frac{BNa}{BF}, \text{ and } g = \frac{FN}{AF} - \frac{FNa}{BF};$$

then $GH^2 = (AB - fy)^2 + g^2y^2$. Moreover, $EG^2 = 1 + y^2$ and $EH^2 = (1+y)^2a^2$. Now, the area of the triangle EGH

$$= \tfrac{1}{4}\,(2\,EG^2EH^2 + 2\,EH^2GH^2 + 2\,GH^2EG^2 - GH^4 - EG^4 - EH^4)^{\frac{1}{2}}.$$

But the area of EGH $= \frac{1}{2}$ (GH . EP), also. Therefore

$$\text{GH . EP} = \tfrac{1}{2}\left(2\,\text{EG}^2\text{EH}^2 + 2\,\text{EH}^2\text{GH}^2 + 2\,\text{GH}^2\text{EG}^2 - \text{GH}^4 - \text{EG}^4 - \text{EH}^4\right)^{\frac{1}{2}},$$

and $4\text{EP}^2\left[(\text{AB}-fy)^2+g^2y^2\right] = 2a^2(1+y^2)^2 + 2a^2(1+y^2)\left[(\text{AB}-fy)^2+g^2y^2\right]$
$+ 2(1+y^2)\left[(\text{AB}-fy)^2+g^2y^2\right] - \left[(\text{AB}-fy)^2+g^2y^2\right]^2 - (1+y^2) - (1+y^2)^2a^4;$

$$\therefore \left[2(1+a^2)(b^2+g^2)-(1-a^2)^2-(b^2+g^2)^2\right]y^4 - 4\text{AB}f\left[(1+a^2)-(b^2+g^2)\right]y^3$$
$$+2\left[(1+a^2)(f^2+g^2)-(1-a^2)^2+(1+a^2)\,\text{AB}^2-(3f^2+g^2)\,\text{AB}^2 - 2(f^2+g^2)\,\text{EP}^2\right]y^2$$
$$-4\text{AB}f\left[(1+a^2)-(\text{AB}^2+2\text{EP}^2)\right]y + \left\{\text{AB}^2\left[2(1+a^2)-(\text{AB}^2+4\text{EP}^2)\right] - (1-a^2)^2\right\} = 0.$$

This equation enables us to find G in any given instance, and then, if we draw a circle on EG as diameter, the reflecting surface is cut in the required point P.

Thus, to give an example, let AE = 1, EB = ·5429, AB = 1·4143, and EP = ·5357; then AF = 1·5143, BF = 1·7286, FN = 1·4429, AN = ·4429, and BN = ·9714, whence $f = \cdot 5976$, $g = \cdot 4780$, and $a = \text{EB} = \cdot 5429$. Our equation then yields the following result, after reduction :—

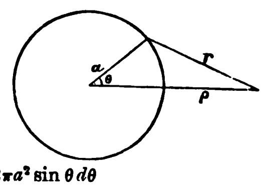
$$y^4 - 3\cdot 5458y^3 - \cdot 2529y^2 + 6\cdot 3971y - 2\cdot 3881 = 0;$$

whence $y = \cdot 415 = \text{AG}$; and $\text{BH} = a\text{AG} = \cdot 2253$. Find G and H by these, join GH, and the perpendicular EP will give P.

[A solution by Dr. Curtis is given on p. 59 of Vol. 39 of *Reprints*.]

7669. (By Professor Townsend, F.R.S.)—A thin uniform spherical shell being supposed to attract, according to the law of the inverse fifth power of the distance, a material particle moving freely in either region of its space external or internal to its mass; if, in either case, the current velocity of the particle be that from infinity under the action of the force, show that its trajectory will be an arc of a circle orthogonal to the surface of the shell.

Solutions by (1) A. H. Curtis, LL.D., D.Sc.; (2) *the* Proposer.



1. If a denote the radius of the sphere, ρ the distance of the centre of the sphere from the attracted point, and r the distance of this point from any particle of the shell, and if V denote the function which, differentiated with regard to x, y, z, will give the corresponding components of attraction,

$$\text{V} = -\tfrac{1}{4}\iint \frac{\mu\,ds}{r^4} = -\tfrac{1}{4}\int_0^{\pi} \frac{\mu\, 2\pi a^2 \sin\theta\, d\theta}{\left[a^2+\rho^2-2a\rho\cos\theta\right]^2}$$
$$= \frac{\mu\pi a}{4\rho}\left\{\frac{1}{(a-\rho)^2} - \frac{1}{(a+\rho)^2}\right\} = \frac{\mu\pi a^2}{(a^2-\rho^2)^2}.$$

As this expression involves only ρ and constants, it shows, as also appears *à priori*, that the total attraction passes through the centre of the sphere, and for the orbit we must have

$$\frac{h^2}{p^2} = \int F\,d\rho = \int \frac{dV}{d\rho}\,d\rho = V = \frac{\mu\pi a^2}{(a^2-\rho^2)^2},$$

no constant being brought in by integration, as the velocity is that from infinity; or the equation of the orbit may be written $\rho^2 \sim a^2 = kp$. This curve will lie in a plane passing through the centre of the sphere and the line of initial velocity of the particle, and is plainly a circle as radius of curvature $\frac{\rho\,d\rho}{dp} = k$, and, when $p = 0$, $\rho = a$, therefore this circle cuts the section of the sphere made by its plane at right angles.

2. The potential of the attraction, for the law of the inverse fifth power of the distance, of a thin uniform spherical shell of mass m and radius a, at the distance r from its centre, being $= -\frac{1}{4}\frac{m}{(r^2-a^2)^2}$, and the square of the velocity from infinity at the same distance under the action of the force being consequently $= \frac{1}{2}\frac{m}{(r^2-a^2)^2}$; therefore, equating the latter to its equivalent $\frac{h^2}{p^2}$ in the trajectory of the particle, we have, for the relation between the p and r of the trajectory,

$$p^2 = 2\frac{h^2}{m}(r^2-a^2)^2 = k^2(r^2-a^2)^2;$$

which represents, as is well known, a circle, the tangential distance of whose circumference from the centre of the shell $= a$, and the reciprocal of whose diameter $= k$; and therefore, &c., as regards the property.

7404. (By Professor Wolstenholme, M.A.)—In a triangle whose sides are of lengths 57613·67, 50178·48, 34134·03, prove that the inscribed circle passes through the centre of the circumscribed circle and through the orthocentre.

Solution by George Heppel, M.A.

In a triangle of the kind suggested, we must have $IO = IP = r$. Now let $\Sigma(\cos A) = u$, $\Sigma(\cos A\cos B) = v$, $\cos A\cos B\cos C = w$. Then

$$r = R(u-1),\quad IO^2 = R^2(3-2u),\quad IP^2 = 4R^2(1-u+v-2w).$$

Also, since $A+B+C = 180°$, $u^2-2v+2w-1 = 0$. These equations give

$$u = \sqrt{2},\quad v = \tfrac{1}{4}(5-2\sqrt{2}),\quad w = \tfrac{1}{4}(3-2\sqrt{2});$$

and cos A, cos B, cos C are the three roots of

$$x^3-\sqrt{2}\,x^2+\tfrac{1}{4}(5-2\sqrt{2})\,x-\tfrac{1}{4}(3-2\sqrt{2}) = 0.$$

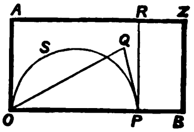
Here $x-\frac{1}{2}$ is a factor, and, solving the remaining quadratic, the three cosines are ·5, ·8080488, ·1061648. The angles are 60°, 36° 5′ 39″·4, 83° 54′ 20″·6. It will be found that the given triangle has its sides in the ratio of the sines of these angles.

[The angles of the triangle are 60°, $60° \pm \alpha$, where $\cos\alpha = \surd 2 - \frac{1}{2}$; the radii of the several circles of the triangle are

$R = 28970{\cdot}56$, $r = 12000$, $r_1 = 63789{\cdot}81$, $r_2 = 40970{\cdot}55$, $r_3 = 23121{\cdot}88$, and the distance between the orthocentre and the circumcentre is 23479·64.]

7603. (By the Editor.)—If on a rectangle AOBZ two random points (P, Q) be taken, P on the base OB, and Q on the surface OZ, show, by a general solution, that, OA remaining constant, (1) as OB increases indefinitely from zero to infinity, the probability that the triangle OPQ is acute-angled decreases from $\frac{1}{2}$ to 0; and (2) in the cases when OB = OA, $OB = \frac{3}{2}OA$, OB = 2OA, OB = 4OA, the probability will fall short of $\frac{1}{2}$ by the approximate values $\frac{22}{168}$, $\frac{33}{168}$, $\frac{44}{168}$, $\frac{68}{168}$, respectively.

Solutions by (I.) D. Biddle; (II.) *the* Proposer.



I. The point Q, in order to form with OP an acute-angled triangle, must be between the parallels AO, RP, and outside the semicircle OSP. The average length of $OP = \frac{1}{2}OB$, and the chance of Q being in the variable space AP (OB being fixed) is also $\frac{1}{2}$. Consequently the chance of Q being in position to form with OP an acute-angled triangle is $\frac{1}{2}$, when the space enclosed by the semicircle vanishes as it does when OB has diminished to zero. When OB is infinitely greater than OA, the semicircle absorbs on the average the whole of the space AP, and leaves no room for Q in the requisite position. Consequently the chance is then 0.

The actual chance at any limit of OB and any position of P is represented by the ratio subsisting between that portion of AP not included by the semicircle, and the entire rectangle AOBZ. And when the whole of the semicircle lies within the rectangle at all positions of P, that is, when OB does not exceed 2AO, the mean area of the semicircle can be deducted from the mean area of AP, and the remainder, in the ratio it bears to the entire rectangle, gives the required chance.

In these instances $OA \,.\, OP - \frac{1}{8}\pi \,.\, OP^2$ gives the area of the space in question for any single position of P, and the mean area for all positions of $P = OB \,.\, OA \left(\frac{1}{2} - \frac{1}{3} \cdot \frac{1}{8}\pi \frac{OB}{OA}\right)$, so that the probability of Q being in this space falls short of $\frac{1}{2}$ by $\cdot 1309 = \frac{21{\cdot}99}{168}$ when OB = OA, by $\cdot 19635 = \frac{32{\cdot}9868}{168}$ when $OB = \frac{3}{2}OA$, and by $\cdot 2618 = \frac{43{\cdot}98}{168}$ when OB = 2OA.

In the fourth instance given, viz., when OB = 4OA, the case is materially altered, since the semicircle (on OP when OP > 2OA) extends beyond the boundaries of the rectangle. However, we have already obtained the mean area of the said space whilst the semicircle is within the rectangle, that is, in the present instance, when $OP < \frac{1}{2}OB$. Therefore we have simply to find the area of the space left by the semicircle, when $OP > \frac{1}{2}OB$.

This area, for any single position of P, is $OP \,.\, OA - (\frac{1}{8}\pi \,.\, OP^2 - \text{segment})$. The segment of the semicircle when $OP = OB = 4OA$ is easily found, since its height $= OA$. The sector is accordingly just $\frac{2}{3}$ of the semicircle and the segment $= \frac{1}{12}\pi \,.\, OP^2 - OA\,(OP^2 - 4OA^2)^{\frac{1}{2}}$. From this it diminishes to zero when $OP = \frac{1}{2}OB$. To the same point, the mean area of the rectangle is $\frac{3}{4}OB \,.\, OA$, and of the semicircle $\frac{5}{7} \,.\, \frac{1}{8}\pi \,.\, OB^2$. Consequently, the mean required space for that portion of the series is

$$OB \,.\, OA \left(\frac{3}{4} - \frac{5}{7} \,.\, \frac{1}{8}\pi \,.\, \frac{OB}{OA}\right) + \text{mean segment.}$$

And the mean required space for the other portion of the series will be

$$OB \,.\, OA \left(\frac{1}{4} - \frac{1}{12} \,.\, \frac{1}{8}\pi \,.\, \frac{OB}{OA}\right).$$

The mean between these two will bear the same ratio to $OB \,.\, OA$ that the required chance bears to unity.

The mean segment can be found from a series ranging from $\frac{1}{4}$ diameter to zero in height, and each multiplied by $4 / (1-2h)^2$, since we have

$$h = (\tfrac{1}{2}OP - OA) / OP, \quad \text{and} \quad OP^2 : OA^2 = 4OA^2 : OA^2(1-2h)^2.$$

This gives a result of $\cdot 6596\, OA^2$ as the mean of the segments extending beyond the rectangle when $OP > \frac{1}{2}OB$. Hence we have

$$OB \,.\, OA \left(\frac{3}{4} - \frac{5}{7} \,.\, \frac{1}{8}\pi \,.\, \frac{OB}{OA}\right) + \cdot 6596 = \cdot 2876,$$

$$OB \,.\, OA \left(\frac{1}{4} - \frac{1}{12} \,.\, \frac{1}{8}\pi \,.\, \frac{OB}{OA}\right) = \cdot 4764,$$

$$\tfrac{1}{2}(\cdot 2876 + \cdot 4764) = \cdot 382 = \text{mean space required for Q.}$$

Now $4 \,.\, \cdot 382 = 168 : 16\cdot 044$; thus the chance is

$$\frac{16\cdot 044}{168} = \frac{1}{2} - \frac{67\cdot 956}{168}.$$

II. *Otherwise* :—The triangle OPQ will be acute-angled, if Q fall anywhere on an area (S, say) contained between the convex circumference of the semicircle OSP, the two tangents OA, PR, and the side AZ of the rectangle. Hence, for every position of P, the probability of an acute-angled triangle will be the ratio of the area (S) to the entire area of the rectangle on which Q must fall. Also the probability of P's falling on any portion of the side OB will be the ratio of that portion to the whole length of OB. Let the breadth OA of the rectangle = unity, its length $OB = 2\lambda$, and $OP = 2x$; then, putting S_1, S_2 for the respective values of S when $x < 1$ and > 1, we have

$$S_1 = 2x - \tfrac{1}{2}\pi x^2, \quad S_2 = 2x - x^2 \operatorname{cosec}^{-1} x - (x^2-1)^{\frac{1}{2}}.$$

Hence, putting p for the probability that the triangle APQ will be acute-angled, and, supposing the length of the rectangle not less than twice its breadth (λ not < 1), we shall have

$$p = \int_0^1 \frac{S_1}{2\lambda}\frac{2dx}{2\lambda} + \int_1^\lambda \frac{S_2}{2\lambda}\frac{2dx}{2\lambda}, \text{ or } 2\lambda^2 p = \int_0^1 S_1\,dx + \int_1^\lambda S_2\,dx \;\ldots\ldots(\alpha).$$

But, if (2) the length does not exceed twice the breadth (λ not > 1), we have only to consider S_1; and then $2\lambda^2 p = \int_0^\lambda S_1\,dx$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\beta)$.

Now we readily find $\int S_1\,dx = x^2 - \frac{1}{6}\pi x^3$, $\int_0^1 S_1\,dx = 1 - \frac{1}{6}\pi$;

$$\int 3S_2\,dx = 3x^2 - 2x\,(x^2-1)^{\frac{1}{2}} - x^3 \operatorname{cosec}^{-1} x + \log_e\left[x + (x^2-1)^{\frac{1}{2}}\right];$$

$$\int_1^\lambda S_2\,dx = \lambda^2 - 1 - \tfrac{2}{3}\lambda\,(\lambda^2-1)^{\frac{1}{2}} + \tfrac{1}{6}\pi - \tfrac{1}{3}\lambda^3 \operatorname{cosec}^{-1}\lambda + \tfrac{1}{3}\log_e\left[\lambda + (\lambda^2-1)^{\frac{1}{2}}\right].$$

Hence, when the length is not less than twice the breadth (λ not >1),

(α) gives $p = \frac{1}{2} - \dfrac{(\lambda^2-1)^{\frac{1}{2}}}{3\lambda} - \frac{1}{6}\lambda \operatorname{cosec}^{-1}\lambda + \dfrac{1}{6\lambda^2}\log_e\left[\lambda + (\lambda^2-1)^{\frac{1}{2}}\right]$ ......(α'),

an expression which, by putting $\lambda = \operatorname{cosec}\alpha$, may be written

$$p = \tfrac{1}{2} - \tfrac{1}{3}\cos\alpha - \tfrac{1}{6}\alpha \operatorname{cosec}\alpha + \tfrac{1}{6}\sin^2\alpha \,.\, \log_e \cot\tfrac{1}{2}\alpha \quad ....(\alpha'').$$

When the length is four times the breadth ($\lambda = 2$, $\alpha = \frac{1}{6}\pi$), then, from equation (α') or (α''), we have

$$p = \tfrac{1}{6}(3-\sqrt{3}) - \tfrac{1}{18}\pi + \tfrac{1}{24}\log_e(2+\sqrt{3}), \text{ or } p = \cdot 09166 = \tfrac{2}{21} \text{ nearly.}$$

When the length is not greater than twice the breadth (λ not >1), equation (β) gives $p = \frac{1}{2} - \frac{1}{12}\lambda\pi$ ....(β').

When the length is equal to twice the breadth ($\lambda = 1$), then, from equation (β'), we have $p = \frac{1}{2} - \frac{1}{12}\pi$, or $p = \frac{5}{21}$ nearly.

When the rectangle is a square ($\lambda = \frac{1}{2}$), then we have

$$p = \tfrac{1}{2} - \tfrac{1}{24}\pi, \text{ or } p = \tfrac{31}{84}, \text{ nearly.}$$

If we suppose the side OB (or 2λ) to increase without limit, the side OD remaining constant, then p decreases without limit, and becomes zero when λ is infinite; and, as OB decreases without limit, p increases up to $\frac{1}{2}$, which is its limit when λ is zero.

7676. (By J. J. Walker, M.A., F.R.S.)—If $F(xyz) = 0$ is the equation to any surface referred to rectangular axes, show that the equation to the curve in which it is cut by the plane $x\cos\alpha + y\cos\beta + z\cos\gamma = p$, referred to the foot of p as origin, and the line in which the plane is cut by that containing the line p and the axis of z, and a line at right angles thereto, as axes, is obtained by substituting for x, y, z, in $F(xyz) = 0$,

$$p\cos\alpha + (y\cos\beta - z\cos\gamma\cos\alpha)\operatorname{cosec}\gamma,$$

$$p\cos\beta - (y\cos\alpha + z\cos\beta\cos\gamma)\operatorname{cosec}\gamma,\ \ p\cos\gamma + z\sin\gamma.$$

[It may readily be verified that these formulæ give, *e.g.*, as the equation to the section of $x^2 + y^2 + z^2 - r^2 = 0$, $y^2 + z^2 - r^2 + p^2 = 0$.]

Solution by W. J. Curran Sharp, M.A.

In a paper "On the Plane Sections of Surfaces, &c.," read before the London Mathematical Society, in December, 1883, I have shown that, if (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be any three points, and if

$\frac{\lambda x_1 + \mu x_2 + \nu x_3}{\lambda + \mu + \nu}$, &c. be substituted for x, &c. in the equation to a surface, the resulting equation in $\lambda\mu\nu$ is the equation, in areal coordinates, to the section of the surface by the plane through (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), these being the vertices of the triangle of reference. Mr. WALKER's origin of plane coordinates (A) is the point $(p\cos\alpha,\ p\cos\beta,\ p\cos\gamma)$, and his axes are the lines $\frac{x}{\cos\alpha} = \frac{y}{\cos\beta} = \frac{p - h\cos\gamma}{\sin^2\gamma} = \rho$,

$$\frac{p\cos\alpha - x}{\cos\beta} = \frac{y - p\cos\beta}{\cos\alpha} = \frac{z - p\cos\gamma}{0} = \sigma,$$

and let the two points B and C, which with A determine the plane, be chosen in these lines, in the negative direction, so that for B $\rho = r$, and for C $\sigma = s$. Then AB $= (r-p)\tan\gamma$ and AC $= s\sin\gamma$, also λ, μ, ν being the areal, and η and ζ the rectangular coordinates of any point in the plane, $\lambda : \mu : \nu = 2\Delta ABC + AB\,.\,\eta + AC\,.\,\zeta : -AC\,.\,\zeta : -AB\,.\,\eta$

$= (r-p)\,s\tan\gamma\sin\gamma + (r-p)\tan\gamma\,.\,\eta + s\sin\gamma\,.\,\zeta : -s\sin\gamma\,.\,\zeta : -(r-p)\tan\gamma\,.\,\eta$

$= (r-p)\,s\sin\gamma + (r-p)\,\eta + s\cos\gamma\,.\,\zeta : -s\cos\gamma\,.\,\zeta : -(r-p)\,\eta.$

Therefore the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_2, z_3) being

$(p\cos\alpha,\ p\cos\beta,\ p\cos\gamma)$, $(r\cos\alpha,\ r\cos\beta,\ p\sin\gamma - r\tan\gamma\sin\gamma)$,

and $(p\cos\alpha - s\cos\beta,\ p\cos\beta + s\cos\alpha,\ p\cos\gamma)$,

$$x = (\lambda x_1 + \mu x_2 + \nu x_3) \div (\lambda + \mu + \nu) = \{[(r-p)s\sin\gamma + (r-p)\eta + s\cos\gamma\,.\,\zeta]\,p\cos\alpha$$
$$- s\zeta\cos\gamma\,.\,r\cos\alpha - (r-p)\,\eta\,(p\cos\alpha - s\cos\beta)\} \div (r-p)\,s\sin\gamma$$
$$= p\cos\alpha + (\eta\cos\beta - \zeta\cos\alpha\cos\gamma)\operatorname{cosec}\gamma,$$

$$y = (\lambda y_1 + \mu y_2 + \nu y_3) \div (\lambda + \mu + \nu)$$
$$= \{[(r-p)\,s\sin\gamma + (r-p)\eta + s\zeta\cos\gamma]\,p\cos\beta - s\zeta\cos\gamma\,.\,r\cos\beta$$
$$- (r-p)\,\eta\,(p\cos\beta + s\cos\alpha)\} \div (r-p)\,s\sin\gamma$$
$$= p\cos\beta - (\eta\cos\alpha + \zeta\cos\beta\cos\gamma)\operatorname{cosec}\gamma,$$

$$z = (\lambda z_1 + \mu z_2 + \nu z_3) \div (\lambda + \mu + \nu)$$
$$= \{[(r-p)\,s\sin\gamma + (r-p)\,\eta + s\zeta\cos\gamma]\,p\cos\gamma - s\zeta\cos\gamma(p\sin\gamma - \nu\tan\gamma\sin\gamma)$$
$$- (r-p)\,\eta\,.\,p\cos\gamma\} \div (r-p)\,s\sin\gamma$$
$$= p\cos\gamma + \zeta\sin\gamma\,;$$

and the metrical properties of the sections of surfaces may be investigated by means of this transformation, as I have attempted, in the paper above mentioned, to study the projective properties by the help of the transformation from which I have derived this.

[A solution may be effected by Quaternions as follows:—

If i, j, k be unit-vectors in the directions of the axes of x, y, z respectively, and $i', j'\ k'$ others in those of p and the plane-axes of $-z'$ and $-y'$, as taken in the question, then

$$i' = \cos\alpha\,.\,i + \cos\beta\,.\,j + \cos\gamma\,.\,k,\quad k = \cos\gamma\,.\,i' - \sin\gamma\,.\,j',$$

therefore $j' = \operatorname{cosec}\gamma\,(\cos\alpha\cos\gamma\,.\,i + \cos\beta\cos\gamma\,.\,j - \sin^2\gamma\,.\,k)$,

therefore
$$k' = i'j' = \operatorname{cosec}\gamma\,[-\cos\gamma\,(\cos^2\alpha + \cos^2\beta - \sin^2\gamma)$$
$$+ (-\cos\beta\sin^2\gamma - \cos\beta\cos^2\gamma)\,i + (\cos\alpha\cos^2\gamma + \cos\alpha\sin^2\gamma)\,j$$
$$+ (\cos\alpha\cos\beta\cos\gamma - \cos\alpha\cos\beta\cos\gamma)\,k]$$
$$= -\cos\beta\operatorname{cosec}\gamma\,.\,i + \cos\alpha\operatorname{cosec}\gamma\,.\,j.$$

Now, if ρ be the vector from the original origin to any point on the given plane, we have

$$\rho = xi + yj + zk = pi' - z'j' - y'k'$$
$$= [p \cos \alpha + (y' \cos \beta - z' \cos \alpha \cos \gamma) \operatorname{cosec} \gamma]\, i$$
$$+ [p \cos \beta - (y' \cos \alpha + z' \cos \beta \cos \gamma) \operatorname{cosec} \gamma]\, j$$
$$+ [p \cos \gamma + z' \sin \gamma]\, k;$$

whence the relations in the question follow at once.]

7619. (By M. JENKINS, M.A.)—Prove that the coefficient of x^n in $\dfrac{1}{(1-x)(1-x^2)(1-x^3)}$, is $\frac{1}{2}[n + \mathrm{R}(\frac{1}{3}n)][1 + \mathrm{E}(\frac{1}{3}n)] + \mathrm{E}\,\frac{1}{6}[6 - \mathrm{R}(\frac{1}{3}n)]$, where $\mathrm{E}\left(\dfrac{n}{p}\right)$ is the integral quotient, and $\mathrm{R}\left(\dfrac{n}{p}\right)$ the remainder, when n is divided by p.

Solution by the PROPOSER.

The required coefficient is the number of indefinite partitions of n into 3 parts, say ${}_nP_3$; and by dividing these into groups whose least element is 0, 1, 2, ... respectively, or by dividing the expansion of $\dfrac{1}{(1-x)(1-x^2)}$ by $1-x^3$ synthetically, it may be shown that

$${}_nP_3 = {}_nP_2 + {}_{n-3}P_2 + {}_{n-6}P_2 + \dots \text{ \&c.}$$

Now ${}_nP_2 = 1 + \mathrm{E}(\frac{1}{2}n)$,

$\therefore\ {}_nP_3 = [1 + \mathrm{E}(\frac{1}{2}n)] + [1 + \mathrm{E}\,\frac{1}{2}(n-3)] + \dots$ repeated $1 + \mathrm{E}(\frac{1}{3}n)$ times
$= 1 + \mathrm{E}(\frac{1}{3}n) + [\mathrm{E}(\frac{1}{2}n) + \mathrm{E}\,\frac{1}{2}(n-6) + \mathrm{E}\,\frac{1}{2}(n-12)$ to $1 + \mathrm{E}(\frac{1}{6}n)$ terms$]$
$+ [\mathrm{E}\,\frac{1}{2}(n-3) + \mathrm{E}\,\frac{1}{2}(n-9) + \dots$ to $1 + \mathrm{E}\,\frac{1}{6}(n-3)$ terms$]$;

whence, writing q for $\mathrm{E}(\frac{1}{6}n)$, r for $\mathrm{R}(\frac{1}{6}n)$,

${}_nP_3 = 1 + 2q + \mathrm{E}(\frac{1}{3}r) + [3q + \mathrm{E}(\frac{1}{2}r) + 3(q-1) + \mathrm{E}(\frac{1}{2}r) + \dots 3(0) + \mathrm{E}(\frac{1}{2}r)]$
$+ [3(q-1) + \mathrm{E}\,\frac{1}{2}(r+3) + 3(q-2) + \mathrm{E}\,\frac{1}{2}(r+3) + \dots 3(0) + \mathrm{E}\,\frac{1}{2}(r+3)$
$+ \mathrm{E}\,\frac{1}{2}(r-3)$, the last term being taken only if $r > 3$, whence it may be denoted by $\mathrm{E}\,\frac{1}{6}(r+1)]$,

$\therefore\ {}_nP_3 = 1 + 2q + \mathrm{E}(\frac{1}{3}r) + 3\,\frac{1}{2}q(q+1) + (1+q)\,\mathrm{E}(\frac{1}{2}r) + 3\,\frac{1}{2}q(q-1)$
$+ q\mathrm{E}\,\frac{1}{2}(r+3) + \mathrm{E}\,\frac{1}{6}(r+1)$
$= 1 + 2q + 3q^2 + q[\mathrm{E}(\frac{1}{2}r) + \mathrm{E}\,\frac{1}{2}(r+3)] + \mathrm{E}(\frac{1}{2}r) + \mathrm{E}(\frac{1}{3}r) + \mathrm{E}\,\frac{1}{6}(r+1)$.

Now $\mathrm{E}(\frac{1}{2}r) + \mathrm{E}\,\frac{1}{2}(r+3) = r+1$; and, by trial of all cases from $r = 0$ to $r = 5$, we may substitute $r + \mathrm{E}\,\frac{1}{6}(6-r)$ for $1 + \mathrm{E}(\frac{1}{2}r) + \mathrm{E}(\frac{1}{3}r) + \mathrm{E}\,\frac{1}{6}(r+1)$,

therefore ${}_nP_3 = 2q + 3q^2 + q(r+1) + r + \mathrm{E}\,\frac{1}{6}(6-r)$
$= (3q+r)(1+q) + \mathrm{E}\,\frac{1}{6}(6-r)$,

which proves the theorem, since $3q + r = \frac{1}{2}(6q + 2r) = \frac{1}{2}[n + \mathrm{R}(\frac{1}{3}n)]$.

7506. (By S. TEBAY, B.A.)—Find (1) the form of a when $x^2 + a$ and $x^2 - a$ are rational squares; also (2) deduce the simple values

$$x = (k-l)^2 + 4l^2, \quad a = 8l(k-3l)(k^2-l^2);$$

and (3) give a neat method (with examples) of calculating corresponding numerical values of x and a when a only is integral.

Solution by G. B. MATHEWS, B.A.; *the* PROPOSER; *and others.*

Let $x^2+a = (x+m)^2$; then $x = \dfrac{a-m^2}{2m}$, and $x^2-a = \dfrac{a^2-6am^2+m^4}{4m^2}$.

Let $a^2-6am^2+m^4 = \left(a-\dfrac{k}{l}m^2\right)$; then $m^2 = \dfrac{2al(k-3l)}{k^2-l}$.

Take $a = \dfrac{2l(k-3l)}{k^2-l^2}b^2$, b being any arbitrary quantity; then

$$m = \frac{2l(k-3l)}{k^2-l^2}b, \text{ and } x = \frac{(k-l)^2+4l^2}{2(k^2-l^2)}b.$$

Take $b = 2(k^2-l^2)$, then $x = (k-l)^2+4l^2$, and $a = 8(k-3l)(k^2-l^2)$.

Again, let $\dfrac{2l(k-3l)}{k^2-l^2} = \dfrac{ps^2}{qt^2}$ be a fraction in its lowest terms; thus

$$a = pq\left(\frac{s}{qt}b\right)^2.$$

Take $b = \dfrac{qt}{s}$; then $a = pq$, and $x = \dfrac{qt^2-ps^2}{2st}$.

Example I.—Let $k = 7$, $l = 2$; then $a = \frac{4}{45}b^2 = \dfrac{1\,.\,2^2}{5\,.\,3^2}b^2$. Thus $p = 1$, $q = 5$, $s = 2$, $t = 3$; therefore $a = pq = 5$, and $x = \frac{41}{12}$; thus the two squares are $(\frac{31}{12})^2$, $(\frac{49}{12})^2$.

II.—Let $k = 5$, $l = 1$; then $a = \frac{1}{6}b^2 = \dfrac{1\,.\,1^2}{6\,.\,1^2}b^2$; and we find, as before, $a = 6$, $x = \frac{5}{2}$, and the squares $(\frac{1}{2})^2$, $(\frac{7}{2})^2$.

III.—Let $k = 9$, $l = 1$; then $a = \dfrac{3\,.\,1^2}{5\,.\,2^2}b^2$; as before, $a = 15$, $x = \frac{17}{4}$, and the squares $(\frac{7}{4})^2$, $(\frac{21}{4})^2$.

[If we assume $x^2+a = (m+n)^2$, $x^2-a = (m-n)^2$, we have $a = 2mn$, $x^2 = m^2+n^2 = \square$; hence we may take $m = (k^2-l^2)$, $n = 2kl$; then

$$x = k^2+l^2, \quad a = 4kl(k^2-l^2).$$

If $l = 2l'$, $k = (k'-l')$, we have $x = (k'-l')^2+4l'^2$,

$$a = 8l'(k'-l')(k'+l')(k'-3l') = 8l'(k'-3l')(k'^2-l'^2).$$

As an example, if $x = 101 = 10^2+1^2$, we may take

$$k = 10, \quad l = 1, \quad a = 40\,.\,99 = 3960.$$

See also solution of Quest. 7468, on p. 119 of Vol. 40 of *Reprints*.]

7598. (By Professor WOLSTENHOLME, M.A., Sc.D.)—1. Circles are drawn with their centres on a given ellipse, and touching (α) the major axis, (β) the minor axis; prove that, if $2a$ be the major axis, and e the eccentricity, the whole length of the arc of the curve envelope of these
eccentricity, the whole length of the arc of the curve envelope of these

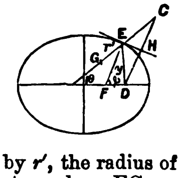
circles is $4a\left(1+\frac{1-e^2}{e}\log\frac{1+e}{1-e}\right)$, $4a\left((1-e^2)^{\frac{1}{2}}+2\frac{\sin^{-1}e}{e}\right)$ ......(α, β).

2. Circles are drawn with their centres on the arc of a given cycloid, and touching (α) the base, (β) the tangent at the vertex; prove that the curve envelope of these circles is (α) an involute of the cycloid which is the envelope of that diameter of the generating circle of the given cycloid which passes through the generating point; (β) a cycloid generated by a circle of radius $\frac{1}{2}a$ rolling on the straight line which is the locus of the centre of the generating circle (radius a) of the given cycloid.

3. Circles are drawn with centres on a given curve and touching the axis of x; prove that the arc of their curve envelope is $x-2\int y\,d\theta$, where x, y are the coordinates of the centre of the circle, and $\frac{dy}{dx}=\tan\theta$.

Solution by A. H. Curtis, LL.D., D.Sc.; Professor Rau, M.A.; *and others.*

The curve, envelope of circles described as in each of the cases included in this question, is by Quetelet's construction (see Salmon's *Higher Curves*) an involute of the caustic by reflexion of the curve due to rays perpendicular to the fixed line which the moving circle touches. In (1), (α) and (β), it is required to find the length of a certain involute of the curve which is the caustic of the ellipse due to rays parallel to an axis.



1. Let the figure represent the ellipse, let DE be any incident ray codirectional with the ordinate y, EF the normal and EG the reflected ray, making respectively with the axis major the angles ϕ and θ, and G be the corresponding point on the caustic; if, then, DC be perpendicular to the tangent at E, C is a point on the envelope sought; denoting EG by r', the radius of curvature of the ellipse by γ, and the chord of curvature along EG, or ED, by c, by the figure,

$$\phi+\phi-\theta=\tfrac{1}{2}\pi, \text{ therefore } \tfrac{1}{2}\pi+\theta=2\phi, \text{ and therefore } d\theta=2d\phi,$$

while, from the formula $\frac{1}{r}+\frac{1}{r'}=\frac{4}{c}$, as $r=\infty$, $r'=\frac{1}{4}c$, therefore, with the usual notation $r'=\frac{1}{4}c=\frac{1}{2}\gamma\sin\phi=\frac{b'^3}{2ab}\frac{py'}{b^2}=\frac{b'^2y'}{2b^2}$.

Now, if ds be the element of the locus of C, we have

$$ds=\text{CG}\,d\theta=(r'+y')\,d\theta, \text{ and } s=8\int_0^{\frac{1}{2}\pi}(r'+y')\,d\theta=8\int_0^{\frac{1}{2}\pi}\left(\frac{b'^2}{2b^2}+1\right)y'\,d\phi,$$

or, as $\frac{\sin\phi}{p}=\frac{y'}{b^2}$,

$$s=8\int_0^{\frac{1}{2}\pi}\frac{b'^2+2b^2}{2p}\sin\phi\,d\phi=4\int_0^{\frac{1}{2}\pi}\left(\frac{a^2b^2}{p^3}+\frac{2b^2}{p}\right)\sin\phi\,d\phi$$

$$=4\int_0^{\frac{1}{2}\pi}\frac{a^2b^2\sin\phi\,d\phi}{(a^2\cos^2\phi+b^2\sin^2\phi)^{\frac{3}{2}}}+8\int_0^{\frac{1}{2}\pi}\frac{b^2\sin\phi\,d\phi}{(a^2\cos^2\phi+b^2\sin^2\phi)^{\frac{1}{2}}}$$

$$=4a^2b^2\int_0^{\frac{1}{2}\pi}\frac{\frac{1}{2}d(\tan^2\phi)}{(a^2+b^2\tan^2\phi)^{\frac{3}{2}}}-8b^2\int_0^{\frac{1}{2}\pi}\frac{d\cos\phi}{[b^2+(a^2-b^2)\cos^2\phi]^{\frac{1}{2}}}$$

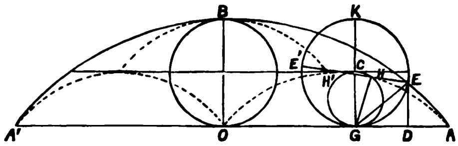
$$=4a^2\left(\frac{1}{a}\right)+4a\left(\frac{1-e^2}{e}\right)\log\left(\frac{1+e}{1-e}\right), \text{ by the formula}$$

$$\int \frac{dz}{(1+z^2)^{\frac{1}{2}}} = \log[z+(1+z^2)^{\frac{1}{2}}], \quad \text{or} \quad s = 4a\left[1+\frac{1-e^2}{e}\log\left(\frac{1+e}{1-e}\right)\right].$$

The discussion of (2) is similar, the difference in the form of the result from case (1) being due to the fact that the formula now employed is

$$\int \frac{dz}{(1-z^2)^{\frac{1}{2}}} = \sin^{-1} z.$$

2. This depends upon the fact that the caustic by reflexion of a cycloid due to rays perpendicular to its base consists of two cycloids as in figure,



each touching the given cycloid, and having for base one half of its base, and that this caustic is the envelope of the diameter referred to. This may be shown geometrically thus :—

Let ABA′ be the given cycloid, DE any incident ray, GEK the generating circle in the corresponding position, and C its centre, draw GH perpendicular to ECE′, and on GC as diameter describe a circle GHCH′, then ∠DEG = ∠CGE = ∠CEG, and therefore CE is the reflected ray and the radius through E of the generating circle in the corresponding position, and arc GH, subtending ∠GCH at circumference of circle GHC = arc GE, subtending same angle at the centre of the circle GEK, therefore = line GA; hence the locus of the point H is a cycloid to which HG is a normal, CE a tangent, and AO the base; thus (2, α) is proved. (2, β) appears from the fact that B is obviously a point on the locus, and it is therefore evident from the figure that the *particular* involute, in this case, is the cycloid stated in the question.

7699. (By R. Knowles, B.A., L.C.P.)—Prove that in any triangle

$$\frac{\cos A}{c\sin B}+\frac{\cos B}{a\sin C}+\frac{\cos C}{b\sin A}=\frac{1}{R} \quad \text{.....................(1).}$$

Solution by Maurice Pontonnier; J. Brill, B.A.; *and others.*

On a $\quad b\sin A = \dfrac{ab}{2R},\quad c\sin B = \dfrac{bc}{2R},\quad a\sin C = \dfrac{ac}{2R};$

donc (1) vient $\quad \dfrac{2R\cos A}{bc}+\dfrac{2R\cos B}{ac}+\dfrac{2R\cos C}{ab}=\dfrac{1}{R};$

d'où : $\quad a\cos A + b\cos B + c\cos C = \dfrac{abc}{2R^2};$

mais $\quad \cos A = \dfrac{b^2+c^2-a^2}{2bc},\quad \cos B = \text{......};$

on a alors : $$-a^4-b^4-c^4+2a^2b^2+2a^2c^2+2b^2c^2=\frac{a^2b^2c^2}{R^2},$$

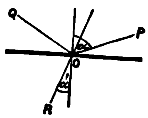
ou $$4b^2c^2\sin^2 A=\frac{a^2b^2c^2}{R^2}, \quad \text{ou} \quad \frac{a}{\sin A}=2R,$$

formule connue, donc (1) est vérifiée.

$$\left[\frac{\cos A}{c\sin B}+\ldots+\ldots=\frac{a\cos A+\ldots}{2\Delta}=\frac{a\cos A+b\cos B+c\cos C}{a\,.\,R\cos A+b\,.\,R\cos B+c.\,R\cos C}=\frac{1}{R}.\right]$$

7602. (By Professor HUDSON, M.A.)—A ray proceeding from a point P, and incident on a plane surface at O, is partly reflected to Q and partly refracted to R: if the angles POQ, POR, QOR be in arithmetical progression, show that the angle of incidence is $\cot^{-1}\left(\frac{\mu-2}{\mu\sqrt{3}}\right)$.

Solution by C. MORGAN, B.A.; J. A. OWEN, B.Sc.; *and others.*



Let α, α' be the angles of incidence and refraction; then $POQ=2\alpha$, $POR=\pi-\alpha+\alpha'$, $QOR=\pi-\alpha-\alpha'$, and, since these angles are in arithmetical progression, we have $2\alpha'=3\alpha-\alpha'-\pi$, and $\alpha-\alpha'=\frac{1}{3}\pi$. But, if μ be the index of refraction, $\sin\alpha=\mu\sin\alpha'=\mu\sin(\alpha-60^\circ)$, whence the stated result follows.

4904 & 6884. (By Dr. HART.)—Find the equation of the Cayleyan of the cubic $x^2y+y^2z+z^2x+2mxyz=0$, and compute the invariants of this cubic.

Solution by W. J. CURRAN SHARP, M.A.

For this form of the equation to a cubic, the Hessian, the discriminant of the polar conic

$$yx'^2+zy'^2+xz'^2+2(mx+y)\,y'z'+2(my+z)\,z'x'+2(mz+x)\,x'y'=0$$

is $$m^2\left[x^2y+y^2z+z^2x+2mxyz\right]-(x^3+y^3+z^3-3xyz)=0,$$

and therefore, when this meets the curve,

$$(x+y+z)(x^2+y^2+z^2-yz-zx-xy)=0,$$

an equation which represents one set of inflexional axes, of which one is real and the other two imaginary, and, since the discriminant $\frac{1}{3}(3^2-6m+4m^2)^2$ of the binary cubic determining the real inflexions is positive, the cubic cannot be cuspidal. In other cases the reduction to this form may be effected by identifying the inflexional axes with the above lines, and the curve is referred to such axes that (y, z), (z, x), (x, y) lie upon the curve, and each is the tangential of the one after it in cyclical order, and therefore its own third tangential. The Cayleyan is the condition that $\alpha x+\beta y+\gamma z=0$ should cut the conics

$$U_1\equiv z^2+2myz+2xy,\quad U_2\equiv x^2+2yz+2mzx,\quad U_3\equiv y^2+2zx+2mxy,$$

in points in involution, *i.e.*,

$$\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma-3m\left(\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha\right)+2m^2\left(\alpha\beta^2+\beta\gamma^2+\alpha^2\gamma\right)-4m^3\alpha\beta\gamma=0.$$

The invariants are

$$S\equiv-(m^4+3m)\quad\text{and}\quad T\equiv-(8m^6+36m^3+27)$$

(and therefore the cubic cannot be cuspidal); the discriminant is $27(8m^3+27)$. So that, if the cubic be nodal, $m^3=-\frac{27}{8}$ and T is positive, *i.e.* (*Quarterly Journal*, Vol. xvi., p. 192) it is crunodal.

7427. (By Professor TOWNSEND, F.R.S.)—A lamina, setting out from any arbitrary position and moving in any arbitrary manner, being supposed to return to its original position after any number of complete revolutions in its plane; show that—

(*a* All systems of points of the lamina which describe curves of equal area in the plane lie on circles fixed in the lamina;

(*b*) All systems of lines of the lamina which envelope curves of equal perimeter in the plane are tangents to circles fixed in the lamina;

(*c*) The two systems of circles, for different values of the area in the former case and of the perimeter in the latter case, are concentric, and have a common centre in the lamina.

Solution by G. B. MATHEWS, B.A.

The motion is determined by the rolling of a curve in the lamina upon a curve in the plane; thus the results at once follow from KEMPE's and McCAY's theorem [given on pp. 82 to 86 and p. 101 of MINCHIN's *Uniplanar Kinematics*].

7574. (By Professor WOLSTENHOLME, M.A., Sc.D.)—If we denote by $F(x,n)$, the determinant of the nth order

$$\begin{vmatrix} x, & 1, & 0, & 0, & 0, & \ldots\ldots \\ 1, & x, & 1, & 0, & 0, & 0\ \ldots \\ 0, & 1, & x, & 1, & 0, & 0\ \ldots \\ \ldots & \ldots & \ldots & \ldots & & \\ 0, & 0, & 0 & \ldots\ldots & 1, & x,\ 1 \\ 0, & 0, & 0 & \ldots\ldots & 0, & 1,\ x \end{vmatrix}$$

prove that $F(x,2r+1)\equiv xF(x^2-2,r)$,

$$F(x,2r)\equiv F(x^2-2,r)+F(x^2-2,r-1),$$

$$F(x,n)\equiv\left(x-2\cos\frac{\pi}{n+1}\right)\left(x-2\cos\frac{2\pi}{n+1}\right)\times\left(x-2\cos\frac{3\pi}{n+1}\right)\ldots\left(x-2\cos\frac{n\pi}{n+1}\right).$$

Solution by B. HANUMANTA RAU, M.A.; Prof. NASH, M.A.; *and others.*

By expanding the determinant, it is easily seen that

$$F(x,n)-xF(x,n-1)+F(x,n-2)=0\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and, therefore, $F(x, n)$ is the coefficient of y^n in the expansion of $(1-xy+y^2)^{-1}$ in ascending powers of y. Let $x = 2\cos\theta$ and $\frac{\pi}{n+1} = a$,

then $$(1-2\cos\theta, y+y^2)^{-1} = (1-ye^{\theta c})^{-1}(1-ye^{-\theta c})^{-1}$$

$$= \frac{e^{\theta c}(1-ye^{-\theta c})-e^{-\theta c}(1-ye^{\theta c})}{(e^{\theta c}-e^{-\theta c})(1-ye^{\theta c})(1-ye^{-\theta c})} = \frac{1}{\sin\theta}[\sin\theta + x\sin 2\theta + x^2\sin 3\theta + \dots],$$

$$\therefore\quad F(x, n) = \frac{\sin(n+1)\theta}{\sin\theta} = 2^n \sin(\theta+a)\sin(\theta+2a)\dots\sin(\theta+na)\dots(2).$$

But $$\sin(\theta+a)\sin(\theta+na) = \sin(\theta+a)\sin(a-\theta)$$

$$= \sin^2 a - \sin^2\theta = \cos^2\theta - \cos^2 a = (\cos\theta-\cos a)(\cos\theta-\cos na),$$

and so on. Therefore

$$F(x, n) = 2^n(\cos\theta-\cos a)(\cos\theta-\cos 2a)\dots(\cos\theta-\cos na)$$

$$= (x-2\cos a)(x-2\cos 2a)\dots(x-2\cos na).$$

If $n = 2r+1$, then $\cos(r+1)a = \cos\frac{1}{2}\pi = 0$,

and $(x-2\cos a)(x-2\cos na) = (x^2-4\cos^2 a) = (x^2-2-2\cos 2a)$,

$$\therefore F(x, 2r+1) = x(x^2-2-2\cos 2a)(x^2-2-2\cos 4a)\dots$$

$$= x\left(x^2-2-2\cos\frac{\pi}{r+1}\right)\left(x^2-2-2\cos\frac{2\pi}{n+1}\right)\dots = xF(x^2-2, r).$$

Again, from (1),

$$F(x, 2r+1) - xF(x, 2r) + F(x, 2r-1) = 0,$$

$$\therefore xF(x, 2r) = F(x, 2r+1) + F(x, 2r-1) = xF(x^2-2, r) + xF(x^2-2, r-1),$$

therefore $$F(x, 2r) = F(x^2-2, r) + F(x^2-2, r-1).$$

7410. (By W. J. C. Sharp, M.A.)—If $N : D$ be a fraction in its lowest terms, and $D \equiv 2^h . 5^k . a^l . b^m . c^n \dots$, where a, b, c, &c. are prime numbers, the equivalent decimal will consist of h or k non-recurring figures (according as h or k is greatest), and of a recurring period, the number of figures in which is a measure of $a^{l-1}(a-1) . b^{m-1}(b-1) . c^{n-1}(c-1)\dots$

Solution by George Heppel, M.A.

Let the equivalent decimal have p non-recurring and q recurring figures. Then $N : D = K : 2^p . 5^p . 10^{q-1}$. Hence, obviously, p must be equal to the highest index of either 2 or 5 in the factors of D. Also, supposing D to have but one other prime factor a, then, from Fermat's theorem, the maximum value q can have is $a-1$. If q has any smaller value, then, since in actual division we have remainder 0 after every q nines used, and we have remainder 0 after using $(a-1)$ nines, therefore q measures $a-1$. Now, if D contains a factor a^l, suppose that the resulting period is one of c digits, and let $10^c = C$. Then l measures $C-1$, therefore it also measures the following:

$$(a-1)(c-1)+a \quad\text{or}\quad (a-1)C+1,$$

$$(a-2)(C^2-C)+(a-1)C+1 \quad\text{or}\quad (a-2)C^2+C+1,$$

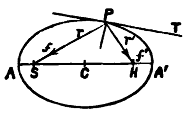
$$(a-3)(C^3-C^2)+(a-2)C^2+C+1 \quad\text{or}\quad (a-3)C^3+C^2+C+1.$$

Proceeding in this way, we see that l measures $C^{a-1}+C^{a-2}+\ldots+C+1$; therefore it measures C^a-1 or $10^{ac}-1$, or q repeated ac times. Hence, the maximum period for a being $(a-1)$, for a^2 it is $a(a-1)$; for a^3 it is $a^2(a-1)$, and so on. Consequently the greatest possible period for D, as given in the question, must be $a^{l-1}(a-1).b^{m-1}(b-1).c^{n-1}(c-1)\ldots$, and from the reasoning given above, the actual period always measures the maximum period.

7657. (By J. CROCKER.)—If an ellipse be described under a force f to focus S and f_1 to focus H, and SP $= r$, HP $= r_1$; prove that

$$\frac{df_1}{dr_1}-\frac{df}{dr}=2\left(\frac{f}{r}-\frac{f_1}{r_1}\right).$$

Solution by D. EDWARDES; W. G. LAX, B.A.; *and others.*



Resolving along the tangent, we have

$$v\frac{dv}{ds}=(f'-f)\frac{dr}{ds} \text{ or } v\frac{dv}{dr}=f'-f \ldots\ldots(1).$$

Again, $\dfrac{v^2}{\rho}=(f+f')\sin SPT=(f+f')\dfrac{b}{(rr')^{\frac{1}{2}}}$,

that is, $$v^2=(f+f')\frac{rr'}{a}.$$

Differentiating this with respect to r, and remembering that $dr'+dr=0$,

$$2(f'-f)=\frac{rr'}{a}\left(\frac{df}{dr}-\frac{df'}{dr'}\right)+\frac{f+f'}{a}(r'-r),$$

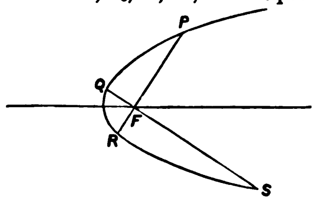
by (1). Putting $r+r'$ for $2a$ and reducing, we have the stated result.

[In a paper by Dr. CURTIS "On Free Motion under the action of *several* Central Forces" (*Messenger of Mathematics*, New Series, No. 109, May 1880), this question is discussed, as a subordinate case, and the condition $\dfrac{1}{r^2}\dfrac{d}{dr}(Fr)-\dfrac{1}{r'^2}\dfrac{d}{dr'}(F'r')=0$, deduced, which is equivalent to the above.]

7547. (By R. TUCKER, M.A.)—PFR, QFS, are two orthogonal focal chords of a parabola, and circles about PFQ, QFR, RFS, SFP cut the axis in points the ordinates to which meet the curve in P′, Q′, R′, S′: prove (1) locus of centres of mean position of P, Q, R, S is a parabola, (latus rectum $\frac{1}{4}$L); (2) Σ (FP′) + 2L = 2Σ (FP); and (3) if also normals at three of the points P, Q, R, S cointersect, then $y_4^{-1}\,\Sigma^3\,y^{-1}) = -24L^{-2}$.

Solutions by the PROPOSER.

Let am_1^2, $2am_1$, &c., be the coordinates of P, Q, R, S; then equation to PQ is



$(m_1+m_2)\,y-2x = 2am_1m_2$......(1),

with condition

$(1-m_1^2)(1-m_2^2)+4m_1m_2 = 0$...(2).

The equation to PR is

$(m_1+m_3)\,y-2x = 2am_1m_3$......(3),

and it is a focal chord, $\therefore\ m_1m_3 = -1$,

similarly $m_2m_4 = -1$ .... (4);

(1) $2\bar{y} = a\,(m_1+m_2+m_3+m_4)$, $4\bar{x} = a\Sigma m^2$,

and $\Sigma\,(m_rm_s) = -6$ .... (5).

therefore $4\bar{y}^2 = 4a\,(\bar{x}-3a)$, that is, $\bar{y}^2 = a\,(\bar{x}-3a)$;

hence locus of mean centres is a parabola whose latus rectum $= \frac{1}{4}$L.

(2) The equation to circle round PFQ is

$$x^2+y^2-a\,(m_1^2+m_2^2)\,x-2a\,(m_1+m_2)\,y+a^2m_1m_2\,(4+m_1m_2) = 0,$$

therefore abscissa of $P' = am_1m_2\,(4+m_1m_2)$ and $FP' = a\,(m_1^2+m_2^2)$ by (2);

therefore $\Sigma\,(FP')+2L = 2a\,[4+\Sigma m^2] = 2\Sigma\,(FP)$.

(3) The equation to the normal through any point (x, y) is

$am^3+(2a-x)\,m-y = 0$, therefore $\Sigma\,(m) = 0$ for P, Q, R,

and $\dfrac{1}{y_4}\Sigma_1^3\left(\dfrac{1}{y}\right) = \Sigma_1^4\dfrac{1}{y_ry_s}-\Sigma_1^3\dfrac{1}{y_ry_s} = -\dfrac{6}{4a^2}$ by (5) $= -24L^{-2}$.

Or, (1) thus:—The equation to PR is $y = k\,(x-a)$,

therefore $x_1+x_3 = 2a\left(1+\dfrac{2}{k}\right)$, $y_1+y_3 = \dfrac{4a}{k}$.

Similarly $x_2+x_4 = 2a\,(1-2k^2)$, $y_2+y_4 = -4ak$,

therefore $4\bar{y} = \Sigma y = 4a\left(\dfrac{1}{k}-k\right)$, *i.e.*, $\bar{y} = a\left(\dfrac{1}{k}-k\right)$,

$$\bar{x} = a\left(1+k^2+\frac{1}{k^2}\right);$$

therefore $\dfrac{\bar{y}^2}{a^2} = k^2+\dfrac{1}{k^2}-2 = \dfrac{\bar{x}}{a}-3$,

i.e., $\bar{y}^2 = a\,(\bar{x}-3a)$.

[This is also the locus of the intersection of two orthogonal normals (see SMITH's *Conics*, p. 104).]

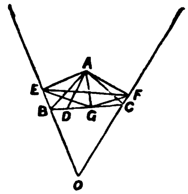
7658. (By S. CONSTABLE.)—The vertex of a triangle is fixed, the vertical angle given, and the base angles move on two parallel straight lines; construct the triangle when the base passes through a fixed point.

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.

For greater generality, let the lines not be parallel, let A be the fixed vertex and OBE, OCF the fixed straight line on which B and C, the extremities of base, move, D being the fixed point on base; draw AE, AF, AG perpendicular, respectively, to OB, OC, BC, then

$\angle$ EGF = AGE + AGF = ABE + ACF = BOC + BAC,

and therefore known; hence one locus of G is a segment of a circle on EF containing this known angle, while another locus is a circle on AD as diameter; the intersection of the two loci gives point G, and determines the required base BC.



7476. (By D. EDWARDES.)—If $xyz = (2-x)(2-y)(2-z)$, show that

$$I \equiv \int_0^1 \int_0^1 xyz\,dx\,dy = \frac{\pi^2}{6} - \frac{5}{4}.$$

Solution by G. B. MATHEWS, B.A.; SARAH MARKS; *and others.*

From the equation, and then by putting $1-x$ for x, we have

$$I = \int_0^1 \int_0^1 \frac{(2-x)(2-y)\,xy}{2-x-y+xy}\,dx\,dy = \int_0^1 \int_0^1 \frac{(1-x^2)(1-y^2)}{1+xy}\,dx\,dy$$

$$= \int_0^1 (1-x^2)\,dx \int_0^1 \left(-\frac{y}{x} + \frac{1}{x^2} + \frac{1-\frac{1}{x^2}}{1+xy}\right) dy$$

$$= \int_0^1 dx\,(1-x^2) \left\{-\frac{1}{2x} + \frac{1}{x^2} + \left(\frac{1}{x} - \frac{1}{x^3}\right)\log(1+x)\right\}$$

$$= \int_0^1 dx\,(1-x^2) \left\{-\frac{1}{2x} + \frac{1}{x^2} + \left(\frac{1}{x} - \frac{1}{x^3}\right)(x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \dots)\right\}$$

$$= \int_0^1 dx\,(1-x^2) \left\{1 - \tfrac{1}{2}x + \tfrac{1}{3}x^2 - \tfrac{1}{4}x^3 + \dots - \tfrac{1}{3} + \tfrac{1}{4}x - \tfrac{1}{5}x^2 + \dots\right\}$$

$$= 2\int_0^1 dx\,(1-x^2) \left\{\frac{1}{1.3} - \frac{x}{2.4} + \frac{x^2}{3.5} - \frac{x^3}{4.6} + \dots\right\}$$

$$= 4\left(\frac{1}{1^2.3^2} - \frac{1}{2^2.4^2} + \frac{1}{3^2.5^2} - \frac{1}{4^2.6^2} + \dots\right).$$

$$= \left(\frac{1}{1^2} + \frac{1}{3^2} - \frac{2}{1.3}\right) - \left(\frac{1}{2^2} + \frac{1}{4^2} - \frac{2}{2.4}\right) + \left(\frac{1}{3^2} + \frac{1}{5^2} - \frac{2}{3.5}\right) - \&c.$$

$$= \frac{\pi^2}{6} - \frac{5}{4}.$$

7666. (By Professor HAUGHTON, F.R.S.)—Prove the following formula for finding the Moon's parallax in altitude in terms of her true zenith distance, viz., $\sin p = \sin P \sin z + \frac{1}{2}\sin^2 P \sin 2z + \frac{1}{3}\sin^3 P \sin 3z + \&c.$

Solution by D. EDWARDES; J. A. OWEN, B.Sc.; *and others.*

If Z be the observed zenith distance,

$$\sin p = \sin P \sin Z \quad \text{or} \quad \sin p = \sin P \sin (z+p),$$

therefore
$$e^{2ip} = \frac{1-\sin P\, e^{-iz}}{1-\sin P\, e^{iz}}.$$

Taking the logarithms, and expanding the right-hand member, we have

$$p = \sin P \sin z + \tfrac{1}{2}\sin^2 P \sin 2z + \tfrac{1}{3}\sin^3 P \sin 3z + \&c.$$

Now, if p be very small, $\sin p = p$ approximately.

7564. (By D. EDWARDES.)—If the sides, taken in order, of a quadrilateral inscribed in one circle, and circumscribed about another, are a, b, c, d; prove that the angle between its diagonals is $\cos^{-1}\frac{ac \sim bd}{ac+bd}$.

Solution by J. A. OWEN, B.Sc.; R. KNOWLES, B.A.; *and others.*

Let h, k be the diagonals, A the angle between b and c, and θ the required angle; then the area of the quadrilateral is $\frac{1}{2}hk\sin\theta = \frac{1}{2}(ad+bc)\sin A$, $\therefore \sin\theta = \frac{ad+bc}{ac+bd}\sin A$; but $\cos A = \frac{b^2+c^2-a^2-d^2}{2(bc+ad)}$;

hence, remembering that $c+a = b+d$, we have

$$\sin^2 A = \frac{4abcd}{(ad+bc)^2}; \text{ hence } \cos\theta = \left\{1 - \frac{4abcd}{(ac+bd)^2}\right\}^{\frac{1}{2}}, \ \cos\theta = \frac{ac \sim bd}{ac+bd}.$$

7575. (By Professor WOLSTENHOLME, M.A., Sc.D.)—Two normals at right angles to each other are drawn respectively to the two (confocal) parabolas $y^2 = 4a(x+a)$, $y^2 = 4b(x+b)$; prove that the locus of their common point is the quartic

$$2y = (a^{\frac{1}{2}} + b^{\frac{1}{2}})\left[x - 2(ab)^{\frac{1}{2}}\right]^{\frac{1}{2}} + (a^{\frac{1}{2}} - b^{\frac{1}{2}})\left[x + 2(ab)^{\frac{1}{2}}\right]^{\frac{1}{2}},$$

which may be constructed as follows:—draw the two parabolas

$$y^2 = (a+b)x - 4ab \pm 2(ab)^{\frac{1}{2}}(x-a-b),$$

and let a common ordinate perpendicular to the axis meet these parabolas in P, p, Q, q, respectively, then the quartic bisects PQ, Pq, pQ, pq. Also the area included beween the quartic and its one real bitangent is $\frac{8}{3}a^2m^2(m+1)(m-1)^3$, where $a = bm^4$, and $a > b$. These results will only be real when ab is positive, or when the two confocals have their concavities in the same sense, but in all cases the rational equation of the quartic is $(y^2-ax+2ab)(y^2-bx+2ab)+ab(a-b)^2=0$.

[The quartic is unicursal, but has only one node at a finite distance ($x=a+b$, $y=0$); there is singularity at ∞, equivalent to two cusps. The class number is 4, and the deficiency 0, so that $2\delta+3\kappa=8$, $\delta+\kappa=3$, or $\delta=1$, $\kappa=2$.]

Solution by B. H. RAU, M.A.; Professor NASH, M.A.; *and others.*

The equations to the normals to the confocal parabolas $y^2=4a(x+a)$, $y^2=4b(x+b)$, are respectively

$$y=mx-a(m^3+m),\quad y=m_1x-b(m_1^3+m_1)\ldots\ldots\ldots\ldots\ldots(1,\ 2).$$

If these are at right angles to each other, $m_1=-\dfrac{1}{m}$, and therefore (2) becomes

$$y=-\frac{x}{m}+b\left(\frac{1}{m^3}+\frac{1}{m}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (1) − (3), $\quad x\left(m+\dfrac{1}{m}\right)-am^2\left(m+\dfrac{1}{m}\right)-\dfrac{b}{m^2}\left(m+\dfrac{1}{m}\right)=0;$

therefore $\quad am^2+\dfrac{b}{m^2}=x;$

therefore $\quad a^{\frac12}m+\dfrac{b^{\frac12}}{m}=[x+2(ab)^{\frac12}]^{\frac12}$, and $a^{\frac12}m-\dfrac{b^{\frac12}}{m}=[x-2(ab)^{\frac12}]^{\frac12}$.

Eliminating x from (1) and (3), we have $y=\dfrac{b}{m}-am$,

$$\therefore\quad 2y=2\left(\frac{b}{m}-am\right)=(a^{\frac12}+b^{\frac12})[x-2(ab)^{\frac12}]^{\frac12}+(a^{\frac12}-b^{\frac12})[x+2(ab)^{\frac12}]^{\frac12},$$

since the radicals may be taken with either the positive or negative sign.

The quartic may be constructed with the help of the curves

$$y=(a^{\frac12}+b^{\frac12})[x-2(ab)^{\frac12}]^{\frac12}\ \text{ and }\ y=(a^{\frac12}-b^{\frac12})[x+2(ab)^{\frac12}]^{\frac12},$$

which are the parabolas given by the equation

$$y^2=(a+b)x-4ab\pm 2(ab)^{\frac12}(x-a-b).$$

Squaring the equation to the quartic, we have

$$4y^2=(a+b+2a^{\frac12}b^{\frac12})(x-2a^{\frac12}b^{\frac12})+(a+b-2a^{\frac12}b^{\frac12})(x+2a^{\frac12}b^{\frac12})+2(a-b)(x^2-4ab)^{\frac12};$$

or $\quad [2y^2-(a+b)x+4ab]^2=4(a-b)^2x^2-16ab(a-b)^2,$

or $\quad (y^2-ax+2ab)(y^2-bx+2ab)+ab(a-b)^2=0,$

which is the rational equation to the quartic.

7287 & 7353. (By Professor WOLSTENHOLME, M.A., D.Sc.)—(7278.) Two circles have radii a, b, the distance between their centres is c, and $a>b+c$; prove that, (1) if any straight line be drawn cutting both circles, the ratio of the squares of the segments made by the circles has the minimum value

$$a\left\{[(a+b)^2-c^2]^{\frac12}+[(a-b)^2-c^2]^{\frac12}\right\}:b\left\{[(a+b)^2-c^2]^{\frac12}-[(a-b)^2-c^2]^{\frac12}\right\};$$

and (2) the distances of the straight line corresponding to this minimum from the centres of the two circles will be in the same ratio.

(7353.) Prove that the maximum and minimum values of

$$u \equiv \frac{a^2 - x^2}{b^2 - (x - c\cos\theta)^2},$$

where x, θ are both variable, a, b, c are given positive constants, and $a > b + c$ are the roots of the quadratic $u^2b^2 - u(a^2 + b^2 - c^2) + a^2 = 0$.

Solution by D. Edwardes.

(7353.) The conditions $\frac{du}{dx} = 0$, $\frac{du}{d\theta} = 0$ lead to $x = u(x - c\cos\theta)$, $\sin\theta\,(x - c\cos\theta) = 0$. Also the sign of $\frac{d^2u}{dx^2}\frac{d^2u}{d\theta^2} - \left(\frac{d^2u}{dx\,d\theta}\right)^2$ is the same as that of $u\cos 2\theta$. If $x - c\cos\theta = 0$, then $x = 0$, $\theta =$ odd multiple of $\frac{1}{2}\pi$, and u is positive, so that this solution must be rejected. We have then $\sin\theta = 0$ and $u = \frac{x}{x \pm c}$. Either sign gives the same quadratic for u, by the elimination of x, viz., $b^2u^2 - u(a^2 + b^2 - c^2) + a^2 = 0$.

Moreover, both roots of this equation are positive, so that they are real maximum and minimum values, since $u\cos 2\theta$ is then positive. The minimum value is

$$\frac{a^2 + b^2 - c^2 - \left[\{(a+b)^2 - c^2\}\{(a-b)^2 - c^2\}\right]^{\frac{1}{2}}}{2b^2}$$

or

$$\frac{a}{b}\,\frac{[(a+b)^2 - c^2]^{\frac{1}{2}} - [(a-b)^2 - c^2]^{\frac{1}{2}}}{[(a+b)^2 - c^2]^{\frac{1}{2}} + [(a-b)^2 - c^2]^{\frac{1}{2}}}.$$

(7278.) The foregoing value solves this Question, a, b being the radii of the circles, and x the perpendicular from the centre of the larger circle upon the cutting line. And $u = \frac{x}{x - c\cos\theta}$ or $\frac{x}{x - c}$ when u is a minimum by the above work, θ being the inclination of the cutting line to the line joining the centres. But $\frac{x}{x - c\cos\theta}$ is the ratio of the perpendiculars in the Question.

7446. (By R. Knowles, B.A., L.C.P.)—(Suggested by Question 7385.)—In an equilateral triangle ABC a circle is inscribed, and a tangent to the circle meets the sides CB, CA in the points A′, B′; the line joining the orthocentre of the triangle A′B′C with the centre of its circumscribing circle meets BC or AC in D; prove that, in either case, as A′B′ varies, the maximum and minimum values of DC are respectively two-ninths and two-thirds of a side of the equilateral triangle.

Solution by G. Heppel, M.A.; G. B. Mathews, B.A.; *and others.*

Let $a =$ side of triangle; $CA' = x$; $CB' = y$; $A'B' = u$.

Then $$u^2 = (a - x - y)^2 = x^2 - xy + y^2;$$

whence $y = \dfrac{a(a-2x)}{2a-3x}$. In triangle A′B′C,

$R = \dfrac{u}{\sqrt{3}}$; $\sin A' = \dfrac{y\sqrt{3}}{u}$; $\sin B' = \dfrac{x\sqrt{3}}{u}$; $\cos A' = \dfrac{2x-y}{2u}$; $\cos B' = \dfrac{2y-x}{2u}$.

Hence, with C as origin and CB as axis of x, the orthocentre is $\left\{\dfrac{y}{2}, \dfrac{2x-y}{2\sqrt{3}}\right\}$, and the circumcentre is $\left\{\dfrac{x}{2}, \dfrac{2y-x}{2\sqrt{3}}\right\}$. Forming the equation of the joining line and putting $y = 0$, we have

$$DC = \frac{x+y}{3} = \frac{a^2-3x^2}{3(2a-3x)}.$$

This is a maximum or minimum when $x = \frac{1}{3}a$, or a, and these give $\frac{2}{9}a$ and $\frac{2}{3}a$ as the values of DC. The first must be a maximum and the second a minimum, for, when $x = 0$ or $\frac{1}{2}a$, $DC = \frac{1}{6}a$; and when $x = \frac{2}{3}a$ or ∞, $DC = \infty$.

7499. (By R. Tucker, M.A.)—OA, OB are two fixed lines, A is a fixed peg, and B a peg movable along OB; an inextensible endless string, passing round AB, is kept stretched by a pencil C; find the envelope of the loci of the curves traced out by C, on the plane AB, by varying the position of B.

Solution by Arthur Hill Curtis, LL.D., D.Sc.

Let the length of the loop be $2l$; then, as the distance of a focus of a conic from the remote extremity of the major axis is half the sum of the major axis and the distance between the foci, it follows that, if with A as centre and radius l a circle be drawn, this circle will touch, at the remote extremity of its axis major, each of the ellipses defined; consequently it is the envelope required.

7401. (By R. Russell, B.A.)—Find (1) $A_1, A_2, A_3 \ldots A_{2n+1}$, such that $A_1(x-a_1)^{2n+1} + A_2(x-a_2)^{2n+1} + \ldots + A_{2n+1}(x-a_{2n+1})^{2n+1}$

$$\equiv P(x-a_1)(x-a_2)\ldots(x-a_{2n+1});$$

and (2) show that A_r is an invariant of the equation whose roots are the quantities $a_1, a_2, \ldots a_{2n+1}$ leaving out a_r.

Solution by the Proposer.

The answer is

$$I_1\Delta_1(x-a_1)^{2n+1} + I_2\Delta_2(x-a_2)^{2n+1} + \ldots = P(x-a_1)(x-a_2)\ldots(x-a_{2n+1}),$$

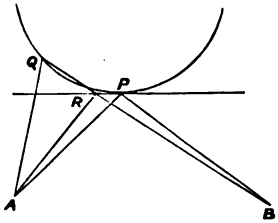
where Δ_r = product of differences of all the roots leaving out a_r, I_r = the invariant formed for the same roots which corresponds to

$$\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\epsilon-\zeta)^2$$

for a sextic, and P = the product of differences of $a_1 a_2 \ldots a_{2n+1}$.

7734. (By J. CROCKER.)—If A and B are fixed points; find, on a fixed circle, a point P such that AP + PB is a minimum.

Solutions by B. H. RAU, M.A.; Dr. CURTIS; *and others.*



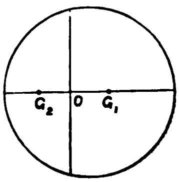
This is another form of Questions 6878, 7422, 7653, for, if P be such that PA and PB make equal angles with the tangent to the circle at the point P, and we take any point Q on the circle, and join QA, QB, the latter meeting the tangent at R, then we have

$$AQ + QB > AR + RB > AP + PB,$$

hence AP + PB is a minimum. [See *Reprint*, Vol. 39, p. 59, and Vol. 41.]

5522. (By Professor ASAPH HALL, M.A.)—If a planet be spherical and ϕ be the angle at the planet between the Earth and the Sun, and a the radius of the sphere; prove that the distance of the centroid of the planet's apparent disk from its true centre will be $\frac{8a}{3\pi}\sin^2\frac{1}{2}\phi$ when the planet is gibbous, and $\frac{8a}{3\pi}\cos^2\frac{1}{2}\phi$ when the planet is crescent.

Solution by the Rev. T. C. SIMMONS, M.A.



Take the case of the planet being gibbous; we have then to investigate the centroid of a figure consisting of a semicircle radius a joined to an ellipse whose semi-axes are a and $a\cos\phi$. Let O be true centre of disc, G_1 and G_2 the centroids of the semicircle and semi-ellipse; then $OG_1 = \frac{4a}{3\pi}$, $OG_2 = \frac{4a\cos\phi}{3\pi}$; therefore if $\bar{x}$ denote distance from O of centroid of apparent disc

$$\bar{x} = \frac{\frac{4a}{3\pi}\cdot\frac{\pi a^2}{2} - \frac{4a\cos\phi}{3\pi}\cdot\frac{\pi a^2\cos\phi}{2}}{\frac{1}{2}\pi a^2 + \frac{1}{2}\pi a^2\cos\phi} = \frac{4a}{3\pi}\cdot\frac{1-\cos^2\phi}{1+\cos\phi} = \frac{8a}{3\pi}\sin^2\frac{\phi}{2}.$$

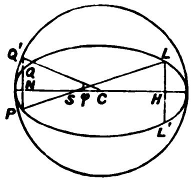
When the planet is crescent, we obtain

$$\bar{x} = \frac{4a}{3\pi}\cdot\frac{1-\cos^2\phi}{1-\cos\phi} = \frac{8a}{3\pi}\cos^2\frac{\phi}{2}.$$

7683. (By R. TUCKER, M.A.)—LSP and LHL′ are a focal chord and a latus rectum respectively of an ellipse, and the circle LL′P cuts the curve again in Q (ϕ); prove that $\tan^2\frac{1}{2}(\phi) = (1+e)^3/(1-e)^3$.

Solution by W. G. LAX, B.A.; J. S. JENKINS; *and others.*

The circle through LPL′ cuts the curve again in Q, where QNP is perpendicular to the major axis. Produce NQ to cut the auxiliary circle in Q′, and join Q′ to C, then ∠ Q′CH = ϕ. Let a be (x, y); then, by combining the equations to the ellipse and to L′Q, we get



$$x = -\frac{ae(3+e^2)}{3e^2+1}, \quad \tan^2 \tfrac{1}{2}\phi = \frac{1-\cos\phi}{1+\cos\phi} = \frac{(1+e)^3}{(1-e)^3}.$$

3873. (By J. B. SANDERS.)—The horizontal section of a cylindrical vessel is 100 square inches, its attitude is 36 inches, and it has an orifice whose section is $\frac{1}{10}$ of a square inch; find in what time, if filled with a fluid, it will empty itself, allowing for the contraction of the vein.

Solution by A. MARTIN, M.A.; Prof. EVANS, M.A.; *and others.*

Put $100 = \text{K}$, $36 = h$, $\frac{1}{10} = k$, v = velocity, and x = altitude of the surface of the fluid at any time t.

Then (WALTON's *Problems, Hydrodynamics*, p. 142),

$$v = \left(\frac{2gx}{1-\frac{k^2}{\text{K}^2}}\right)^{\frac{1}{2}}, \quad \text{and} \quad kvdt = -\text{K}dx \quad \text{...............(1, 2).}$$

Therefore
$$dt = -\frac{\text{K}dx}{dv} = -\frac{\text{K}\left(1-\frac{k^2}{\text{K}^2}\right)^{\frac{1}{2}}dx}{k(2gx)^{\frac{1}{2}}} \quad \text{......................(3).}$$

Integrating (3) between the limits $x=0$, $x=h$, we have

$$t = \frac{2\text{K}\left(1-\frac{k^2}{\text{K}^2}\right)^{\frac{1}{2}}h^{\frac{1}{2}}}{k(2g)^{\frac{1}{2}}} = \frac{2\left(\frac{\text{K}^2}{k^2}-1\right)^{\frac{1}{2}}h^{\frac{1}{2}}}{(2g)^{\frac{1}{2}}} = 11 \text{ minutes } 36{\cdot}5 \text{ seconds,}$$

taking the coefficient of efflux = 0·62.

7696. (By ALPHA.)—Two guns are fired at a railway station at an interval of 21 minutes, but a person in a train approaching the station observes that 20 minutes 14 seconds elapse between the reports; supposing that sound travels 1125 feet per second, show that the velocity of the train is 29·064 miles per hour.

Solution by B. Reynolds, M.A.; W. J. Greenstreet, B.A.; *and others.*

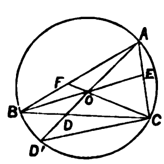
Space traversed by train during the interval between the hearings of the two reports = 46 × 1125 ft.; hence the velocity of train

$$= (46 \times 1125) \div 1214 \text{ ft. per second} = 29{\cdot}064 \text{ miles per hour.}$$

7660. (By R. Knowles, B.A., L.C.P.)—From the angular points of a triangle ABC, lines are drawn through the centre of the circum-circle to meet the opposite sides in D, E, F, respectively; prove that

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{2}{R}.$$

Solution by W. G. Lax, B.A.; J. S. Jenkins; *and others.*



Let O be the circum-centre; produce AOD to D′, and join CD′; then we have

$$\angle BAD = BCD' = \tfrac{1}{2}\pi - C;$$

$$AD = \frac{c \sin ABD}{\sin ADB} = \frac{c \sin B}{\sin(B + 90^\circ - C)}$$

$$= \frac{c \sin B}{\cos(C-B)} = \frac{2R \sin C \sin B}{\cos(C-B)};$$

$$\frac{1}{AD} = \frac{\cos(C-B)}{2R \sin C \sin B} = \frac{1}{2R}(1 + \cot B \cot C),$$

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{1}{2R}[3 + \Sigma(\cot B \cot C)] = \frac{1}{2R}(3+1) = \frac{2}{R}.$$

[In the solution to Question 7594 (p. 80 of this volume), it is proved that

$$\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1; \text{ hence } \frac{AD-R}{AD} + \frac{BE-R}{BE} + \frac{CF-R}{CF} = 1,$$

wherefrom the result immediately follows.]

7633. (By Professor Genese, M.A.)—A circle is inscribed in a segment of a circle containing an angle θ: the point of contact with the base divides it into segments h, k. Prove that (1) the radius of the inscribed circle is $\dfrac{hk}{h+k}\cot\tfrac{1}{2}\theta$; and hence (2) that the inscribed circle of a triangle touches the nine-point circle.

I. *Solution by* G. Heppel, M.A.; J. A. Owen, B.Sc.; *and others.*

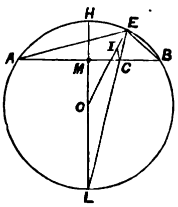
(1) Let AB be the base of the segment, O and P the centres, and c and r the radii of the original and inscribed circles. Draw PC and OD perpendicular to AB, then $AC = h$, $CB = k$, $AD = c\sin\theta$, $OD = \pm c\cos\theta$, $h + k = 2c\sin\theta$. Also $PO^2 = (c\sin\theta - h)^2 + (r - c\cos\theta)^2 = (c-r)^2$, whence we obtain the stated result.

(2) In any triangle ABC, where b is $>c$, let A′ be the mid-point of BC D the foot of the perpendicular from A, and H the point of contact of the inscribed circle with BC. Then $BD = \frac{1}{2}a - \frac{b^2-c^2}{2a}$; $BH = \frac{1}{2}a - \frac{1}{2}(b-c)$; $BA' = \frac{1}{2}a$. And these are in order of magnitude, for $b+c > a$, and therefore H is always between D and A′. Applying the result just obtained to the segment cut off by DA′ from the nine-point circle, whose radius is $\frac{1}{2}R$, we have $\sin\theta = \frac{1}{2}DA' \div \frac{1}{2}R = \frac{DA'}{R} = \frac{b^2-c^2}{2aR} = \frac{\sin^2 B - \sin^2 C}{\sin A} = \sin(B-C)$; hence $\theta = B - C$; also $h = \frac{1}{2}(b-c)$; $k = \frac{b^2-c^2}{2a} - \frac{b-c}{2} = \frac{s_1(b-c)}{2a}$; whence the radius of the circle touching the nine-point circle, and touching BC in H, is $s_1(b-c)\cot\frac{1}{2}(B-C) / (b+c) = s_1 \tan\frac{1}{2}A = r$.

II. *Solution by* MORGAN JENKINS, M.A.

Professor GENESE's result may be put in the following geometrical form:—

(1) If a circle be inscribed in a given segment, the diameter of this circle is equal to the product of the segments of the chord divided by the height of the supplementary segment.

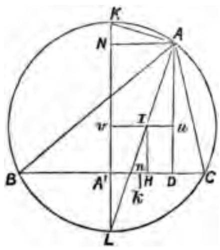


If I be the centre of the in-circle touching the segment at E and the chord AB at C; and H, M, L, the mid-points of the arc of the given segment, the chord, and the supplementary arc, respectively; then EC passes through L, and EI through O, the centre of the circle AHBL;

and $$\frac{IC}{OL} = \frac{EC}{LE} = \frac{EC \cdot CL}{LE \cdot LC} = \frac{AC \cdot CB}{LM \cdot LH};$$

therefore $$IC = \frac{1}{2}\frac{AC \cdot CB}{ML}.$$

(2) Let I be the in-centre of the triangle ABC, KA′L the diameter perpendicular to BC, A′ being the mid-point of BC, IH, AD perpendiculars on BC, AN parallel to BC, vIu parallel to BC cutting KL in v, AD in u.



Then $Iu \cdot Iv : AN^2 = IA \cdot IL : LA^2$
$= KL \cdot r : LN \cdot LK$,

$Iu \cdot Iv : LN \cdot KN = KN \cdot r : LN \cdot KN$;

therefore $Iu \cdot Iv = KN \cdot r$.

Now, the centroid of the triangle ABC being the internal centre of similitude of the circumcircle and the nine-points circle, $NK = 2nk$, where nk is the height of the segment $A'kD$ of the nine-points circle, on the opposite side of A′D to I. Hence $\frac{A'H \cdot DH}{2nk} = r$, and therefore the in-circle touches the nine-points circle.

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